

The supercritical regime in the normal matrix model with cubic potential

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Abstract

The normal matrix model with a cubic potential is ill-defined and it develops a critical behavior in finite time. We follow the approach of Bleher and Kuijlaars to reformulate the model in terms of orthogonal polynomials with respect to a Hermitian form. This reformulation was shown to capture the essential features of the normal matrix model in the subcritical regime, namely that the zeros of the polynomials tend to a number of segments (the motherbody) inside a domain (the droplet) that attracts the eigenvalues in the normal matrix model.

In the present paper we analyze the supercritical regime and we find that the large n behavior is described by the evolution of a spectral curve satisfying the Boutroux condition. The Boutroux condition determines a system of contours Σ_1 , consisting of the motherbody and whiskers sticking out of the domain. We find a second critical behavior at which the original motherbody shrinks to a point at the origin and only the whiskers remain.

In the regime before the second criticality we also give strong asymptotics of the orthogonal polynomials by means of a steepest descent analysis of a 3×3 matrix valued Riemann-Hilbert problem. It follows that the zeros of the orthogonal polynomials tend to Σ_1 , with the exception of at most three spurious zeros.

1 Introduction

The normal matrix model is a probability measure on the space of $n \times n$ normal matrices M of the form

$$\frac{1}{Z_n} e^{-\frac{n}{t} \operatorname{Tr}(MM^* - V(M) - \overline{V}(M^*))} dM, \quad t > 0,$$

with a given potential function V and $\overline{V}(z) = \overline{V(\overline{z})}$. In the limit as $n \rightarrow \infty$ the eigenvalues of M fill out a two-dimensional domain $\Omega = \Omega(t)$, called the droplet, see Figure 1, whose boundary evolves according to Laplacian growth (also known as Hele-Shaw flow) as the time parameter t increases. See [27, 28] and the surveys [19, 29].

The eigenvalues of M are a determinantal point process that is analyzed by polynomials $P_{k,n}$, $\deg P_{k,n} = k$, that are orthogonal with respect to the inner product

$$\langle f, g \rangle = \iint_{\mathbb{C}} f(z) \overline{g(z)} e^{-\frac{n}{t}(|z|^2 - V(z) - \overline{V(z)})} dA(z) \quad (1.1)$$

where dA denotes Lebesgue measure on the complex plane, see [10]. The inner product varies with n . The zeros of the diagonal polynomials $P_{n,n}$ do not fill out the droplet, but instead are believed to cluster on certain one-dimensional arcs inside Ω . These arcs are referred to as the motherbody or skeleton. The orthogonal polynomials were analyzed under various situations cases in [3, 10, 15].

In the interesting case where V is a polynomial of degree ≥ 3 , the above has to be modified since the integrals do not converge. This is done by Elbau and Felder [10, 11] by using a cut-off, or by Ameur, Hedenmalm and Makarov [1, 2] by a modification of V outside the droplet. These approaches work for t small enough, and in fact up to a critical time t_* when the boundary of the droplet develops one or more cusps. At the same time the motherbody meets the boundary at the cusp.

After criticality the Laplacian growth breaks down, but it may be continued in a weak averaged form as in [22, 23]. A main feature of the supercritical regime is the appearance of one-dimensional arcs (we call them whiskers) that point out of the droplet. In the context of [22] these whiskers are interpreted as pressure shock waves. Appearance of the whiskers in the supercritical regime was also observed in [4] (for complex orthogonal polynomials with some exponential weight).

We are going to analyze this phenomenon in the simplest case where $V(z) = \frac{1}{3}z^3$ is a polynomial of degree 3. In this case we want to view the Laplacian growth as the evolution of an algebraic equation (the spectral curve) also after the critical time. Before criticality the spectral curve has genus zero. After criticality the curve has higher genus and it is characterized by the Boutroux condition, which means that all the periods of a certain meromorphic differential are purely imaginary.

We show in the model with a cubic potential that we can select such a curve with the Boutroux condition for $t \in (t_*, t_{**})$ where t_{**} is a finite second critical time. Then the motherbody consists of

$$\Sigma_1 = \Sigma_1^o \cup \Sigma_1^w$$

where Σ_1^o is the part that remains from the original motherbody and Σ_1^w are the whiskers, that (partly) stick out of the droplet Ω . At the second criticality t_{**} we find that both Ω and Σ_1^o disappear, and only the whiskers Σ_1^w remain.

We also follow in this paper the approach of [5] where the orthogonal polynomials are replaced by polynomials that are orthogonal with respect to a certain bilinear form. This bilinear form is well-defined in the cubic model for all time. Before critical time it was shown in [5] that the zeros of the polynomials $P_{n,n}$ tend to the motherbody Σ_1 with a limiting probability distribution μ_1 . See [16] for an extension to higher degree potentials. After criticality we find that the Boutroux condition guarantees the existence of a certain probability measure μ_1 on Σ_1 and we prove that all but at most three zeros of the polynomials $P_{n,n}$ cluster on Σ_1 with μ_1 as limiting zero counting measure. This result further supports the use of the Boutroux condition after criticality. From a different perspective, the Boutroux condition was also suggested in [23], Section 2.2 as a possible way to define the evolution of the Laplacian growth beyond criticality.

We next give a more precise statement of our results.

2 Statement of results

2.1 Spectral curves and Boutroux condition

As it was already stated, we study the model of Laplacian growth in the plane with a cubic potential $V(z) = \frac{1}{3}z^3$. In [5] the more general potential $V(z) = \frac{t_3}{3}z^3$ with $t_3 > 0$ was studied, but it can be reduced to $t_3 = 1$ by a simple scaling.

Before the first criticality t_* the Laplacian growth is governed by the algebraic equation

$$P(\xi, z) = \xi^3 - z^2\xi^2 - (1+t)z\xi + z^3 + A = 0 \quad (2.1)$$

with the t -dependent constant

$$A = A_1(t) = \frac{1}{32} (1 + 20t - 8t^2 - (1 - 8t)^{3/2}), \quad t < t_* = \frac{1}{8}, \quad (2.2)$$

see [5, Lemma 4.3]. The choice of constant (2.2) is dictated by the requirement that the Riemann surface \mathcal{R} associated with (2.1) has genus zero.

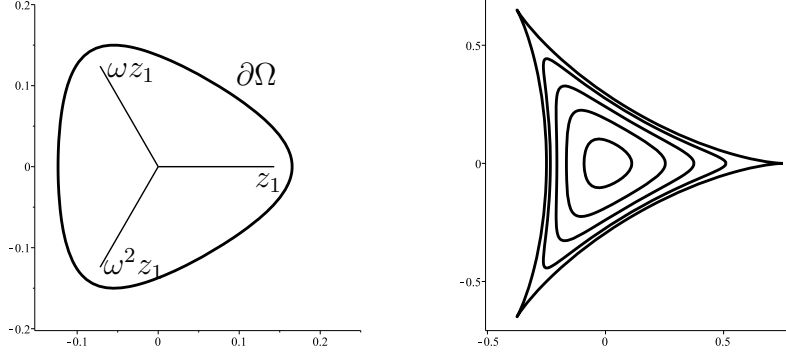


Figure 1: Laplacian growth in the subcritical case. Left panel shows the boundary of the droplet Ω and skeleton Σ_1 for the value $t = 0.02$. Right panel show the growth of $\partial\Omega$ for the values $t = 0.01, 0.04, 0.07, 0.10, 0.125$.

For $A = A_1(t)$, there are three branch points $z_1, \omega z_1, \omega^2 z_1$ and three nodes $z_2, \omega z_2, \omega^2 z_2$ of \mathcal{R} , where $z_2 > z_1 > 0$ and $\omega = e^{2\pi i/3}$. Precise values are $z_1 = \frac{3}{4}(1 - \sqrt{1 - 8t})^{2/3}$ and $z_2 = \frac{1}{4}(3 + \sqrt{1 - 8t})$.

There is a solution $\xi_1(z)$ of (2.1) that behaves like

$$\xi_1(z) = z^2 + tz^{-1} + O(z^{-4}), \quad \text{as } z \rightarrow \infty. \quad (2.3)$$

The solution (2.3) has an analytic continuation to the domain $\mathbb{C} \setminus \Sigma_1$ where

$$\Sigma_1 = [0, z_1] \cup [0, \omega z_1] \cup [0, \omega^2 z_1]. \quad (2.4)$$

Then the equation $\xi_1(z) = \bar{z}$ with $z \in \mathbb{C} \setminus \Sigma_1$ defines the boundary of a domain $\Omega(t)$ that evolves according to the model of Laplacian growth, see [5, Theorem 2.6]. Putting $\xi = \bar{z}$ in (2.1) we find that $\partial\Omega(t)$ is characterized by the equation

$$\partial\Omega(t) : \quad 2 \operatorname{Re}(z^3) - |z|^4 - (1+t)|z|^2 + A_1(t) = 0. \quad (2.5)$$

The boundary $\partial\Omega(t)$ encloses the motherbody Σ_1 , and the nodes $z_2, \omega z_2, \omega^2 z_2$ are exterior to $\partial\Omega(t)$. Note that $z = \omega^j z_2$ for $j = 0, 1, 2$ also satisfies the equation (2.5), but these are isolated points, and are not considered to be part of $\partial\Omega(t)$.

In the cut-off model of Elbau and Felder [10, 11], the eigenvalues in the normal matrix model cluster on $\Omega(t)$ with uniform density. The zeros of the orthogonal polynomials accumulate on the set Σ_1 with a probability measure μ_1 on Σ_1 as the limit of the normalized zero counting measures. The measure has the property that

$$\int_{\Sigma_1} \frac{d\mu_1(s)}{z - s} = \frac{1}{\pi t} \iint_{\Omega} \frac{dA(s)}{z - s}, \quad z \in \mathbb{C} \setminus \Omega.$$

At the critical $t = t_*$ the three branch points $\omega^j z_1$, $j = 0, 1, 2$, come to the boundary $\partial\Omega(t)$, which develops three cusps. Also the node z_2 then coincides with z_1 , see Figure 1.

For the supercritical $t > t_*$, we are still working with a spectral curve of the form (2.1) but with a different determination of $A = A(t)$. The corresponding Riemann surface \mathcal{R} has now genus three, where the branch points of \mathcal{R} can be obtained through the discriminant $D(P)(z) = Q(z^3)$ of (2.1) with respect to the variable ξ (the cubic polynomial $Q(w)$ is given in (3.1)). The evolution is now described by the following condition.

Definition 2.1. *Let $t > 0$ and $A \in \mathbb{R}$. We say that the meromorphic differential ξdz defined on the compact Riemann surface \mathcal{R} associated with the equation (2.1) has the **Boutroux condition** if all the periods of ξdz are purely imaginary. That is,*

$$\oint_{\gamma} \xi dz \in i\mathbb{R} \quad (2.6)$$

for every closed contour γ on \mathcal{R} that avoids the poles of ξdz .

The poles of ξdz are at the two points at infinity, with real residues $\pm t$, see e.g. (2.8). The condition (2.6) is therefore satisfied for contours γ that only go around the poles and are homotopic to zero on \mathcal{R} . In particular the Boutroux condition is satisfied if the Riemann surface has genus zero.

In the supercritical case $t > t_*$, the Riemann surface associated with (2.1) has genus three (unless $A = 0$ in which case the genus is two) and then (2.6) presents a condition on the ξdz periods of the non-trivial cycles γ on the surface.

It turns out that the cubic equation (2.1) has nine branch points, namely $\omega^j z_k$, $j = 0, 1, 2$, $k = 1, 2, 3$ where $\omega = e^{2\pi i/3}$ and z_1, z_2, z_3 are the branch points lying in the sector

$$S_0 = \{z \in \mathbb{C} \mid -\frac{\pi}{3} < \arg z < \frac{\pi}{3}\}. \quad (2.7)$$

In the case of interest we can take z_1 to be real and $z_3 = \bar{z}_2$ with $\text{Im } z_2 > 0$. The three sheeted Riemann surface \mathcal{R} associated with (2.1) has the sheet structure as in Figure 2.

The restriction to the three sheets defines three functions that have the asymptotic behavior

$$\begin{aligned} \xi_1(z) &= z^2 + tz^{-1} + O(z^{-4}), & \text{as } z \rightarrow \infty, \\ \xi_2(z) &= z^{1/2} - \frac{1}{2}tz^{-1} + O(z^{-5/2}), & \text{as } z \rightarrow \infty, z \in S_0, \\ \xi_3(z) &= -z^{1/2} - \frac{1}{2}tz^{-1} + O(z^{-5/2}), & \text{as } z \rightarrow \infty, z \in S_0. \end{aligned} \quad (2.8)$$

The first two sheets are connected by the cuts

$$\Sigma_1 = \Sigma_1^o \cup \Sigma_1^w, \quad \Sigma_1^o = \bigcup_{j=0}^2 [0, \omega^j z_1] \quad (2.9)$$

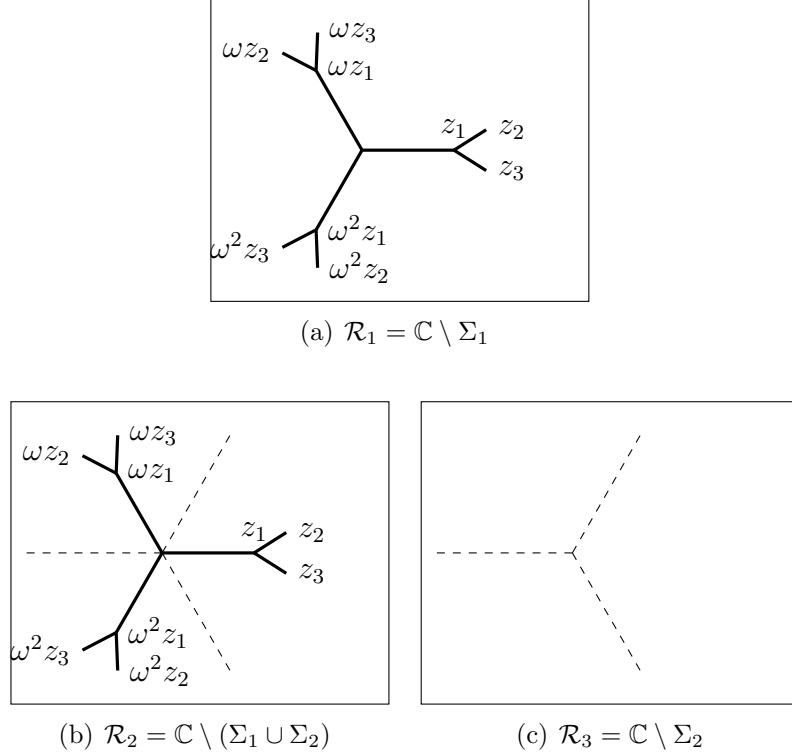


Figure 2: The three sheets \mathcal{R}_1 , \mathcal{R}_2 and \mathcal{R}_3 of the Riemann surface

where Σ_1^w are cuts that connect $\omega^j z_1$ with $\omega^j z_2$ and $\omega^j z_3$ for $j = 0, 1, 2$. The sheets \mathcal{R}_2 and \mathcal{R}_3 are connected via the cut

$$\Sigma_2 = \{z \in \mathbb{C} \mid z^3 \in \mathbb{R}^-\}. \quad (2.10)$$

We refer to Σ_1^w as whiskers that stick out of the branch points $\omega^j z_1$ for $j = 0, 1, 2$, see Figure 3. The whiskers that connect the branch points $\omega^j z_1$, $\omega^j z_2$ and $\omega^j z_3$ for $j = 1, 2, 3$, are arbitrary at this point, but it turns out that they can be defined in a special way if the Boutroux condition is satisfied. This is part of our main result which we state as follows.

Theorem 2.2. (a) *There exist $t_{**} > t_*$ and a unique continuous function $A : [t_*, t_{**}) \rightarrow (0, \infty)$ with $A(t_*) = A_1(t_*) = \frac{27}{256}$ and*

$$\lim_{t \rightarrow t_{**}^-} A(t) = 0 \quad (2.11)$$

such that the differential ξdz has the Boutroux condition for every $t \in (t_, t_{**})$ and $A = A(t)$. Moreover, for every $t \in (t_*, t_{**})$ and $A = A(t)$:*

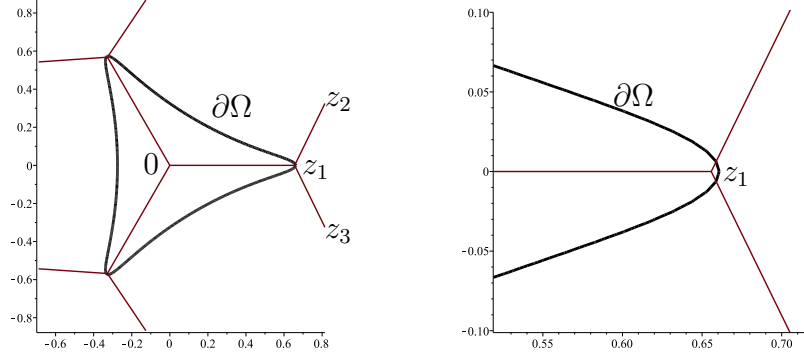


Figure 3: Laplacian growth in the supercritical regime. Left panel shows the domain $\Omega(t)$ and the set $\Sigma_1(t)$ for $t = 0.2$. The right panel zooms in on a neighborhood of z_1 , which lies inside $\Omega(t)$. Most of the whiskers stick out of $\Omega(t)$.

(b) *There is a simple analytic arc $\gamma_{1,2}$ from z_1 to z_2 lying in the sector S_0 such that*

$$\operatorname{Re} \int_{z_1}^z (\xi_1(s) - \xi_2(s)) ds = 0, \quad \text{for every } z \in \gamma_{1,2}. \quad (2.12)$$

We define $\gamma_{1,3} = \overline{\gamma_{1,2}}$. Then $\gamma_{1,3}$ is a simple analytic arc from z_1 to z_3 such that

$$\operatorname{Re} \int_{z_1}^z (\xi_1(s) - \xi_2(s)) ds = 0, \quad \text{for every } z \in \gamma_{1,3}. \quad (2.13)$$

(c) *Let Σ_1 be as in (2.9) with*

$$\Sigma_1^w = \bigcup_{j=0}^2 \omega^j(\gamma_{1,2} \cup \gamma_{1,3}) \quad (2.14)$$

where $\gamma_{1,2}$ and $\gamma_{1,3}$ are as in part (b). Then

$$d\mu_1(s) = \frac{1}{2\pi it} (\xi_{2,+}(s) - \xi_{1,+}(s)) ds, \quad s \in \Sigma_1 \quad (2.15)$$

defines a probability measure on Σ_1 . Here all parts in Σ_1 are oriented outwards, that is, away from the origin, ds is the complex line element that is compatible with this orientation and $\xi_{1,+}$, $\xi_{2,+}$ denote the limiting values of $\xi_{1,2}$ if we approach Σ_1 from the left (and $\xi_{1,-}$, $\xi_{2,-}$ are the limiting values from the right).

(d) *The equation*

$$\partial\Omega(t) : 2 \operatorname{Re}(z^3) - |z|^4 - (1+t)|z|^2 + A = 0 \quad (2.16)$$

defines the boundary of a domain $\Omega(t)$ that is such that the points $\omega^j z_1$, $j = 0, 1, 2$ are inside and the points $\omega^j z_2$, $\omega^j z_3$, $j = 0, 1, 2$, are outside $\Omega(t)$, and

$$\frac{1}{2\pi i} \int_{\partial\Omega(t)} \frac{\bar{s}}{s^k} ds + t \int_{\Sigma_1^w} \frac{d\mu_1(s)}{s^k} = \begin{cases} t & \text{for } k = 0, \\ 1 & \text{for } k = 3, \\ 0 & \text{otherwise.} \end{cases} \quad (2.17)$$

In particular the case $k = 0$ in (2.17) reduces (by Green's theorem) to

$$t = \frac{1}{\pi} \operatorname{area} \Omega(t) + t\mu_1(\Sigma_1^w). \quad (2.18)$$

The proof of Theorem 2.2 is in section 3. The value of t_{**} has been calculated numerically and it is approximately

$$t_{**} = 3.6 \dots$$

The equations (2.17) represent the continuation of the Laplacian growth after criticality. In fact the equations (2.17) also holds before criticality, since then $\Sigma_1^w = \emptyset$ and (2.17) reduces to the equations that characterize the Laplacian growth in terms of the exterior harmonic moments of the droplet [28]. After criticality these equations are modified by the extra term $t \int_{\Sigma_1^w} \frac{d\mu_1(s)}{s^k}$ representing the contribution from the whiskers. See Figure 3 for an illustration.

In the supercritical case, the domain $\Omega(t)$ is no longer growing with t , see Figure 4. In fact, the following corollary shows that $t = t_{**}$ represents a second criticality, see also Figure 5.

Corollary 2.3. *At $t = t_{**}$ we have $z_1(t_{**}) = 0$, whereas $z_2(t_{**})$, $z_3(t_{**})$ are solutions of*

$$z^6 + \frac{t^2 + 20t - 8}{4} z^3 + (1+t)^3 = 0 \quad (2.19)$$

with $t = t_{**}$. Moreover, at $t = t_{**}$ the domain $\Omega(t)$ reduces to a point at the origin.

Proof. To prove Corollary 2.3, it is sufficient to observe that $A(t_{**}) = 0$ reduces: i) the discriminant $Q(w)$, given by (3.1), to

$$Q(w) = w[4w^2 + (t_{**}^2 + 20t_{**} - 8)w + 4(1+t_{**})^3], \quad (2.20)$$

and; ii) the expression (2.16) for $\partial\Omega(t_{**})$ in polar coordinates $z = re^{i\theta}$ to

$$r(r^2 - 2r \cos 3\theta + 1 + t_{**}) = 0, \quad (2.21)$$

with only solution $r = 0$, since the expression in brackets is positive. \square

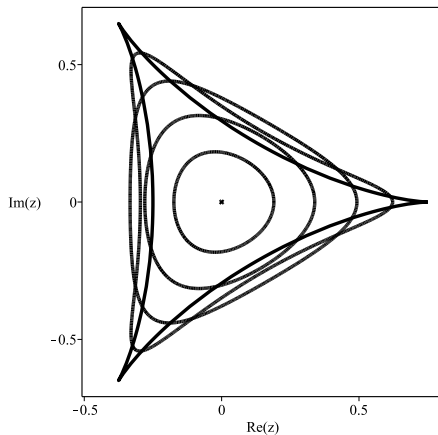


Figure 4: Evolution of the domain $\Omega(t)$ in the supercritical case for the values $t = 0.125, 0.3, 1.0, 2.0, 3.0$ and 3.6 . The domain does not grow with t anymore, but starts to shrink and eventually collapses to a point.

Solving (2.19) we find the following explicit values for $z_{2,3} = z_{2,3}(t_{**})$,

$$z_{2,3} = \frac{1}{2} \left[8 - 20t_{**} - t_{**}^2 \pm i\sqrt{t_{**}(8 - t_{**})^3} \right]^{\frac{1}{3}}.$$

The importance of the Boutroux condition is that it gives rise to the probability measure (2.15) on Σ_1 when $t \in (t_*, t_{**})$, which is a continuation of the probability measure μ_1 on Σ_1 for subcritical $t \in (0, t_*)$ described in [5]. The equations (2.17) may be viewed as characterizing the Laplacian growth after criticality.

The set Σ_2 carries the measure, see [5, formula (4.12)],

$$d\mu_2(s) = \frac{1}{2\pi it} \left(\pm 2s^{1/2} + \xi_{3,+}(s) - \xi_{2,+}(s) \right) ds, \quad s \in \Sigma_2, \quad (2.22)$$

where the choice of sign \pm is such that $\pm 2s^{1/2} + \xi_{3,+}(s) - \xi_{2,+}(s) = O(s^{-1})$ as $s \rightarrow \infty$. As in the subcritical case, the measure μ_2 is real and positive with total mass $1/2$. In [5, Theorems 2.5 and 2.6] the two measures (μ_1, μ_2) were found as the minimizer of a vector equilibrium problem.

This vector equilibrium problem is still relevant in the supercritical case, as it can be

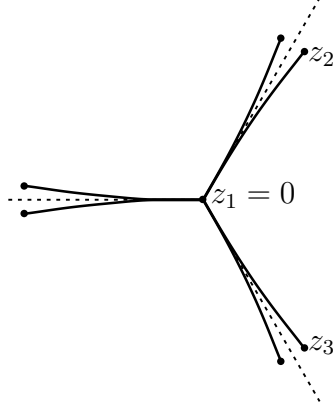


Figure 5: At the second critical time t_{**} , the domain $\Omega(t)$ shrinks to a point, and the set Σ_1 consists of whiskers only, as shown in the figure. The figure also shows the rays $\arg z = \pm\pi/3$, $\arg z = \pi$ that make up the set Σ_2 (dotted lines). The branch points z_2 and z_3 remain in the sector S_0 and do not come to Σ_2 .

shown that the measures jointly minimize the energy functional

$$\begin{aligned}
E(\mu_1, \mu_2) = & \iint \log \frac{1}{|x-y|} d\mu_1(x) d\mu_1(y) + \iint \log \frac{1}{|x-y|} d\mu_2(x) d\mu_2(y) \\
& - \iint \log \frac{1}{|x-y|} d\mu_1(x) d\mu_2(y) \\
& + \frac{1}{t} \sum_{j=0}^2 \int_{\Sigma_1 \cap S_j} \operatorname{Re} \left(\frac{2}{3\sqrt{3}} (\omega^{-j}s)^{3/2} - \frac{1}{3} s^3 \right) d\mu_1(s) \quad (2.23)
\end{aligned}$$

among all vectors of measures (μ_1, μ_2) with μ_j on Σ_j and $\mu_j(\Sigma_j) = \frac{1}{j}$ for $j = 1, 2$. Here $S_j = \omega^j S_0$ for $j = 1, 2$. See [7, 13, 25] for more on (vector) equilibrium problems for logarithmic potentials.

We conjecture that the optimal set Σ_1 is characterized by a max-min property for the energy (2.23). Let \mathcal{T} denote the collection of contours Σ such that:

- (a) Σ is a connected contour (= finite union of analytic arcs) that is symmetric in the real axis, and invariant under the \mathbb{Z}_3 rotational symmetry $z \mapsto \omega z$.
- (b) The part of Σ in S_0 connects 0 to infinity in the directions $\arg z = \pm\frac{\pi}{3}$.

For each $\Sigma \in \mathcal{T}$ we define $\mathcal{E}(\Sigma)$ as the infimum of $E(\mu_1, \mu_2)$ where μ_1 is a measure on

Σ with $\mu_1(\Sigma) = 1$ and μ_2 is a measure on Σ_2 with $\mu_2(\Sigma_2) = \frac{1}{2}$. Then we conjecture that

$$\mathcal{E}(\Sigma_1) = \sup_{\Sigma \in \mathcal{T}} \mathcal{E}(\Sigma).$$

Such a characterization would be analogous to the S -curves that play a role in rational approximation and complex non-Hermitian orthogonality [18, 24].

2.2 Orthogonal polynomials

A basic ingredient in the study of the normal matrix model before criticality are the orthogonal polynomials with respect to the scalar product (1.1), or with respect to a modified version involving a cut-off. The cut-off approach does not work in the supercritical case.

In [5] a different regularization of the scalar product (1.1) was proposed which for the model with cubic potential $V(z) = \frac{1}{3}z^3$ leads to the Hermitian form

$$\langle f, g \rangle = \frac{1}{2\pi i} \sum_{j=0}^2 \sum_{k=0}^2 \epsilon_{j,k} \int_{\Gamma_j} dz \int_{\bar{\Gamma}_k} dw f(z) \bar{g}(w) e^{-\frac{n}{t}(wz - \frac{1}{3}(w^3 + z^3))}, \quad (2.24)$$

defined on polynomials f and g , where Γ_j is an unbounded contour stretching out to infinity from $e^{(2j-1)\pi i/3}\infty$ to $e^{(2j+1)\pi i/3}\infty$ for $j = 0, 1, 2$, and

$$\epsilon_{j,k} = \begin{cases} 0 & \text{if } j = k, \\ 1 & \text{if } j \equiv k + 1 \pmod{3}, \\ -1 & \text{if } j \equiv k - 1 \pmod{3}. \end{cases}$$

The Hermitian form (2.24) satisfies the identity

$$t\langle f, g' \rangle - n\langle zf, g \rangle + n\langle f, V'g \rangle = 0$$

which is also satisfied by (1.1) with the cut-off regularization if one forgets about boundary terms, see [5].

The Hermitian form (2.24) depends on n . The orthogonal polynomial $P_{k,n}$ is a monic polynomial of degree k that satisfies

$$\langle P_{k,n}, z^j \rangle = 0, \quad \text{for } j = 0, 1, \dots, k-1. \quad (2.25)$$

In the subcritical case, it was shown in [5] that the zeros of the diagonal polynomials $P_{n,n}$ accumulate on the motherbody Σ_1 as $n \rightarrow \infty$, with μ_1 as limit of the normalized zero counting measures. This result followed from a steepest descent analysis of

the Riemann-Hilbert (RH) problem that characterizes the orthogonal polynomials with respect to (2.24). The RH problem has size 3×3 . It results in a strong asymptotic formula

$$P_{n,n}(z) = M_{11}(z)e^{ng_1(z)}(1 + O(1/n)), \quad \text{for } z \in \mathbb{C} \setminus \Sigma_1$$

where $g_1(z) = \int \log(z - s)d\mu_1(s)$ and $M_{11}(z)$ is a prefactor that arises as the 11-entry of a global parametrix M that is used in the steepest descent analysis.

The Hermitian form (2.24) also makes sense in the supercritical case, and so does the characterization of the orthogonal polynomials by means of the RH problem. We could do the steepest descent analysis also in this case. The outcome is a strong asymptotic formula for $P_{n,n}$ as in Theorem 2.4, namely

$$P_{n,n}(z) = M_{n,11}(z)e^{ng_1(z)}(1 + O(1/n)), \quad \text{for } z \in \mathbb{C} \setminus \Sigma_1$$

where now the prefactor $M_{n,11}(z)$ varies with n and contains an elliptic theta function, see (2.36) below.

The prefactor $M_{n,11}$ has at most three zeros in $\mathbb{C} \setminus \Sigma_1$ that do not tend to Σ_1 as $n \rightarrow \infty$. They correspond to spurious zeros of $P_{n,n}$ and perform a quasi-periodic motion on the Riemann surface \mathcal{R} . This is the same phenomenon as happens for usual orthogonal polynomials with an orthogonality measure that is supported on several intervals.

In the present case, the quasi-periodic motion takes place on the real part of \mathcal{R} which can be identified with the cycle c_R that starts at z_1 and goes along the intervals $[z_1, \infty)$ and $(-\infty, 0]$ on the first sheet \mathcal{R}_1 , then along $[0, \infty)$ on the third sheet \mathcal{R}_3 , and, finally, from right to left along the interval $(\infty, z_1]$ on the second sheet \mathcal{R}_2 .

For a precise description of $M_{n,11}$ we need some more notions related to the Riemann surface \mathcal{R} , which as we recall, has genus three in the supercritical case $t \in (t_*, t_{**})$. There is a unique holomorphic differential ω_R on \mathcal{R} that is \mathbb{Z}_3 -invariant and that is normalized such that

$$\oint_{c_R} \omega_R = 1. \quad (2.26)$$

It can be explicitly given by

$$\omega_R = \frac{3C}{3\xi^2 - 2z^2\xi - (1+t)z}dz \quad (2.27)$$

with the constant

$$C = \frac{1}{3} \left[\oint_{c_R} \frac{1}{3\xi^2 - 2z^2\xi - (1+t)z} dz \right]^{-1}$$

There is another cycle a_R going around $\Sigma_1^w \cap S_0$ on the first sheet in counterclockwise direction (the cycle passes through the branch point z_1). By symmetry in the real axis we have

$$\tau := \oint_{a_R} \omega_R \in i\mathbb{R}^+ \quad (2.28)$$

The theta function with $q = e^{\pi i(1+\tau)/2}$ (and quasi-period $(1+\tau)/2$) is defined by

$$\theta(s) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2n\pi i s} = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \frac{1+\tau}{2} + 2\pi i n s}. \quad (2.29)$$

It gives an entire function in the complex s -plane with periodicity properties

$$\begin{aligned} \theta(s+1) &= \theta(s) = \theta(-s), \\ \theta(s + \frac{1+\tau}{2}) &= e^{-\pi i \frac{1+\tau}{2}} e^{-2\pi i s} \theta(s), \\ \theta(s + \frac{1-\tau}{2}) &= e^{-\pi i \frac{1-\tau}{2}} e^{2\pi i s} \theta(s). \end{aligned} \quad (2.30)$$

The theta function has a simple zero at the values $s_0 + k + \frac{1+\tau}{2}l$, $k, l \in \mathbb{Z}$, where

$$s_0 = \frac{-1+\tau}{4}, \quad (2.31)$$

and no other zeros.

We further define

$$\beta := \frac{1}{6} \mu_1(\Sigma_1^w) \quad (2.32)$$

and for $\varepsilon > 0$,

$$\mathbb{N}_\varepsilon = \{n \in \mathbb{N} \mid \text{dist}_{\mathbb{R}/\mathbb{Z}} \left(n\beta, \frac{1}{2} + \frac{\tau}{2\pi i} \log 2 + \int_{\infty_1}^{-A^{1/3}} \omega_R \right) \geq \varepsilon \} \quad (2.33)$$

where $A = A(t)$ as before, and $-A^{1/3}$ denotes the point $z = -A^{1/3}$, $\xi = 0$ that is on the first sheet of the Riemann surface, see (2.1).

Theorem 2.4. *Suppose $t \in (t_*, t_{**})$. Let $\varepsilon > 0$. Then for any large $n \in \mathbb{N}_\varepsilon$ the polynomials $P_{n,n}$ exist and*

$$P_{n,n}(z) = (M_{n,11}(z) + O(1/n)) e^{ng_1(z)} \quad \text{as } n \in \mathbb{N}_\varepsilon, \ n \rightarrow \infty \quad (2.34)$$

uniformly for z in compact subsets of $\mathbb{C} \setminus \Sigma_1$, where:

$$g_1(z) = \int_{\Sigma_1} \log(z - s) d\mu_1(s), \quad (2.35)$$

and

$$M_{n,11}(z) = 2^{2 \int_z^{\infty} \omega_R} \frac{\theta(s_0 + \int_{-A^{1/3}}^{-\infty} \omega_R) \theta(s_0 + \int_{-A^{1/3}}^z \omega_R + n\beta - \frac{\tau}{2\pi i} \log 2 - 1/2)}{\theta(s_0 + \int_{-A^{1/3}}^z \omega_R) \theta(s_0 + \int_{-A^{1/3}}^{\infty} \omega_R + n\beta - \frac{\tau}{2\pi i} \log 2 - 1/2)} \times \frac{\xi_1(z)}{(3\xi_1^2(z) - 2z^2\xi_1(z) - (1+t)z)^{1/2}}. \quad (2.36)$$

The error term $O(1/n)$ in (2.34), which depends on ε , is uniform on compact subsets of (t_*, t_{**}) .

The proof of Theorem 2.4 can be found in section 4.

The theta function $\theta(s_0 + \int_{-A^{1/3}}^z \omega_R)$ has zeros at $z = -A^{1/3}$ and also at $-\omega^j A^{1/3}$ for $j = 1, 2$, since by the \mathbb{Z}_3 symmetry of ω_R one has that $\int_{-A^{1/3}}^{-\omega^j A^{1/3}} \omega_R = 0$. However, these zeros in the denominator are cancelled by the fact that the $\xi_1(z)$ has zeros at these same values. If $n \notin \bigcup_{\varepsilon > 0} \mathbb{N}_\varepsilon$ then

$$\theta\left(s_0 + \int_{-A^{1/3}}^{\infty} \omega_R + n\beta - \frac{\tau}{2\pi i} \log 2 - 1/2\right) = 0$$

and then the right-hand side of (2.36) is not well-defined.

For each n there is a point Q_n on the cycle c_R on the Riemann surface, such that

$$\int_{-A^{1/3}}^{Q_n} \omega_R = -n\beta + \frac{\tau}{2\pi i} \log 2 + 1/2 \quad \text{mod } \mathbb{Z}.$$

If Q_n happens to be on the first sheet, say $Q_n = (x_n, \xi_1(x_n)) \in \mathcal{R}_1$, then

$$\theta\left(s_0 + \int_{-A^{1/3}}^{x_n} \omega_R + n\beta - \frac{\tau}{2\pi i} \log 2 - 1/2\right) = 0$$

and so $M_{n,11}(x_n) = 0$ by formula (2.36). By symmetry we then also have $M_{n,11}(\omega^j x_n) = 0$ for $j = 1, 2$. It then follows from the asymptotic formula (2.34) and Hurwitz's theorem from complex analysis that $P_{n,n}$ has a simple zero near each of $\omega^j x_n$, $j = 0, 1, 2$, if $n \in \mathbb{N}_\varepsilon$ is large enough. These are spurious zeros of $P_{n,n}$. The other zeros of $P_{n,n}$ are non-spurious zeros.

Corollary 2.5. *Let $n \rightarrow \infty$ with $n \in \mathbb{N}_\varepsilon$. Then the non-spurious zeros of $P_{n,n}$ tend to Σ_1 and μ_1 is the limit of the normalized zero counting measures.*

Proof. This follows from the asymptotic formula (2.34) and the above consideration on the spurious zeros. Indeed, from (2.34) and (2.35) we have

$$\lim_{\substack{n \rightarrow \infty \\ n \in \mathbb{N}_\varepsilon}} \frac{1}{n} \lim \log |P_{n,n}(z)| = \int_{\Sigma_1} \log |z - s| d\mu_1(s),$$

almost everywhere in $\mathbb{C} \setminus \Sigma_1$, which by standard arguments from logarithmic potential theory, see [25], yields that the non-spurious zeros of $P_{n,n}$ tend to Σ_1 with μ_1 as limiting distribution. \square

2.3 Remark on perturbation analysis around the critical regime

Although the focus of this paper is not on the critical regime, it is interesting to see how the branch points z_j , $j = 1, 2, 3$ approach the cusp point as $\Delta t = t - t_* > 0$ tends to zero, and to compare this with [23].

For the critical values $t = t_* = 1/8$, $A = A_* = 27/256$ the cubic equation (2.1) for the spectral curve has the branch point $z = z_* = 3/4$, $\xi = \xi_* = 3/4$, which is a triple zero of the discriminant of (2.1).

Let us introduce small $\Delta t > 0$ and define $t = t_* + \Delta t$. Direct perturbation analysis of (2.1) indicates the following asymptotic behavior:

$$\begin{aligned} A &= A_* + \frac{9}{16} \Delta t - k(\Delta t)^{3/2} + O((\Delta t)^2), \\ z &= z_* + x(\Delta t)^{1/2} + O(\Delta t), \\ \xi &= \xi_* + x(\Delta t)^{1/2} + \frac{4}{3\sqrt{3}} y(\Delta t)^{3/4} + O((\Delta t)^2), \end{aligned} \tag{2.37}$$

where the constants x, y, k are to be defined. Plugging (2.37) into (2.1) we find

$$\left(y^2 - x^3 - \frac{3}{2}x - k \right) (\Delta t)^{3/2} + O((\Delta t)^{7/4}) = 0,$$

which means that the equation

$$y^2 = x^3 + \frac{3}{2}x + k \tag{2.38}$$

should hold true. The equation (2.38) defines an elliptic Riemann surface that is obtained from the blow up at the criticality.

Recall that the choice of A as a function of t is dictated by the fact that ξdz has the Boutroux condition. Then in the new xy - variables it means that ydx has the Boutroux condition, see Definition 2.1, on the surface (2.38). This is a condition on k in (2.38). The elliptic Riemann surface has one real branch point x_1 and two non-real branch points x_2 and $x_3 = \bar{x}_2$ with $\text{Im } x_2 > 0$. There are non-trivial cycles α and β analogous to the cycles α_0 and β_0 on \mathcal{R} , see Figure 8 below. The period $\oint_{\alpha} ydx$ is always purely imaginary. The mapping $k \mapsto \text{Re}(\oint_{\beta} ydx)$ is strictly increasing (this is analogous to Lemma 3.3 below) and there is a unique value of k with $\text{Re}(\oint_{\beta} ydx) = 0$. This value for k is approximately

$$k = 0.647 \dots \quad (2.39)$$

After k is determined, the branch points x_j , $j = 1, 2, 3$, can be calculated from the cubic equation (2.38) as they are the zeros of $x^3 + \frac{3}{2}x + k$. We find the approximate values

$$x_1 \approx -0.391, \quad x_2 \approx 0.196 + 1.27i, \quad x_3 \approx 0.196 - 1.27i \quad (2.40)$$

which by (2.37) determine approximations for z_j for $j = 1, 2, 3$. In particular

$$z_1 \approx z_* - 0.391(\Delta t)^{1/2} \quad \text{as } \Delta t \rightarrow 0 +. \quad (2.41)$$

Recall from [5, formula (2.19)] that for $t < t_*$ one has $z_1 = \frac{3}{4}(1 - \sqrt{1 - 8t})^{2/3}$ and so if $\Delta t = t - t_* < 0$,

$$z_1 = z_* - \sqrt{2}(-\Delta t)^{1/2} + O(\Delta t) \quad \text{as } \Delta t \rightarrow 0 -. \quad (2.42)$$

In the paper [23] of Lee, Teodorescu and Wiegmann the following equations are given for the motion of the fingertip (formula (28) in [23])

$$e(T) = \begin{cases} -2\sqrt{-T}, & T < 0, \\ -0.553594\sqrt{T}, & T > 0. \end{cases} \quad (2.43)$$

To compare the results one should identify $e(T) = z_1 - z_*$ and $T = \frac{1}{2}\Delta t$. Then $e(T) = -2\sqrt{-T}$ corresponds to the leading behavior in (2.42) for $\Delta t < 0$, and for $T > 0$,

$$e(T) = -0.553594\sqrt{T} = -\frac{0.553594}{\sqrt{2}}(\Delta t)^{1/2} = 0.39145(\Delta t)^{1/2},$$

which corresponds to (2.41).

It is also of interest to consider the intersection point \widehat{z} of $\partial\Omega$ with the positive real line. If we substitute $t = t_* + \Delta t$, $\widehat{z} = z_* + x(\Delta t)^{1/2} + O(\Delta t)$, and the expression for A from (2.37) into (2.16), then we find

$$-\left(x^3 + \frac{3}{2}x + k\right)(\Delta t)^{3/2} + O(\Delta t)^2 = 0.$$

This means that $x = x_1$, where x_1 is the real solution of $x^3 + \frac{3}{2}x + k = 0$. Thus, the distance from \widehat{z} to z_1 is only $O(\Delta t)$ as $\Delta t \rightarrow 0+$, and this is much smaller than the distance from z_1 to the other branch points z_2 and z_3 , which is $O(\Delta t)^{1/2}$. Therefore, in the leading order $O(\Delta t)^{1/2}$, the branch point z_1 is indistinguishable from \widehat{z} . This effect is clearly visible in the left panel of Figure 3.

3 Proof of Theorem 2.2

3.1 The Riemann surface

We start by investigating the Riemann surface that is associated with the cubic equation (2.1). The discriminant of (2.1) with respect to the variable ξ is a polynomial in z of degree 9, which because of the \mathbb{Z}_3 symmetry takes the form

$$D(P)(z) = Q(z^3)$$

with a cubic polynomial $Q(w)$ (these and other calculations were made with the help of Maple)

$$Q(w) = 4w^3 + (t^2 + 20t + 4A - 8)w^2 + (4(1+t)^3 + 18At - 36A)w - 27A^2. \quad (3.1)$$

The discriminant of (3.1) with respect to w has the remarkable factorization

$$D(Q) = 16(t^2 - 7t - 8 + 3A)^3(t(1+t)^3 - 20At + 16A^2 - A), \quad (3.2)$$

which is zero for $A = A_j(t)$, $j = 1, 2, 3$, where $A_1(t)$ is given by (2.2),

$$A_2(t) = \frac{1}{32} (1 + 20t - 8t^2 + (1 - 8t)^{3/2}), \quad t < t_*, \quad (3.3)$$

and

$$A_3(t) = \frac{1}{3}(1+t)(8-t). \quad (3.4)$$

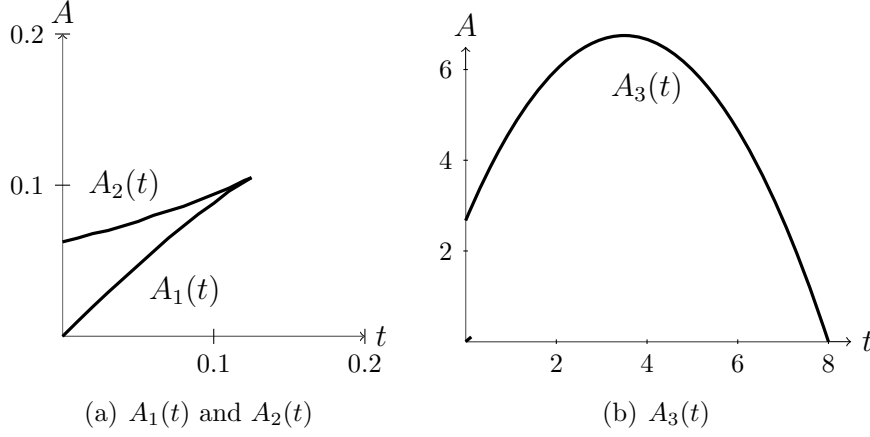


Figure 6: Graphs of $A_1(t)$, $A_2(t)$ and $A_3(t)$. Note the difference in scale between the two plots.

The cubic polynomial (3.1) has three zeros w_1, w_2, w_3 , and at least one of them, say w_1 , is real. If $A \neq 0$ then $w_1 > 0$. Then $D(P)(z)$ has nine zeros, namely

$$z_1 = w_1^{1/3}, \quad z_2 = w_2^{1/3}, \quad z_3 = w_3^{1/3},$$

and their rotations $\omega z_j, \omega^2 z_j$, for $j = 1, 2, 3$, where $z_1 > 0$ is real, and $-\frac{\pi}{3} \leq \arg z_3 \leq 0 \leq \arg z_2 \leq \frac{\pi}{3}$.

If $A = A_j(t)$ for some $j = 1, 2, 3$, then $Q(w)$ has one simple and one double zero. This gives three simple zeros and three double zeros for the discriminant of P . In case $A = A_1(t)$ or $A = A_2(t)$ we have that the double zeros are nodes and there is no branching at these points. Thus, keeping in mind the branch point at infinity, we have only four branch points, which by the Riemann-Hurwitz formula, see e.g. [12, 20], gives that the genus is zero.

For $A = A_3(t)$ the double zeros are branch points located on the rays with angles $\pm\pi/3$ and π , that connect all three sheets. Then the genus is three.

Finally, if $A \in \mathbb{R}^+ \setminus \{A_1(t), A_2(t), A_3(t)\}$ then all zeros are simple zeros of the discriminant of P . Being simple zeros, these give rise to nine branch points of the Riemann surface. Taking note that there is also branching at infinity (between sheets \mathcal{R}_2 and \mathcal{R}_3) we have in total 10 branch points on the Riemann surface when viewed as a three-fold cover of the z -plane. The Riemann-Hurwitz formula then tells us that the genus is three.

At the critical value $t = t_*$, $A = A_* = A_1(t_*)$, we have triple zeros of the discriminant at $z_1, \omega z_1, \omega^2 z_1$ for some $z_1 > 0$. The sheet structure of the Riemann surface is then given by $\mathcal{R}_1 = \mathbb{C} \setminus \Sigma_1$, $\mathcal{R}_2 = \mathbb{C} \setminus (\Sigma_1 \cup \Sigma_2)$, $\mathcal{R}_3 = \mathbb{C} \setminus \Sigma_3$ where $\Sigma_1 = \bigcup_j [0, \omega^j z_1]$ and

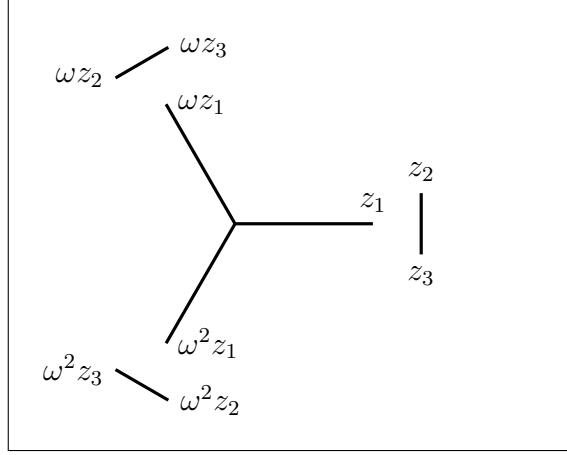


Figure 7: The first sheet \mathcal{R}_1 after deformation of the whiskers Σ_1^w to arcs joining $\omega^j z_2$ to $\omega^j z_3$ disjoint from $\omega^j z_1$, for $j = 0, 1, 2$.

$\Sigma_3 = \{z \in \mathbb{C} \mid z^3 \in \mathbb{R}^-\}$. The sheets \mathcal{R}_1 and \mathcal{R}_2 are connected via Σ_1 , and \mathcal{R}_2 and \mathcal{R}_3 are connected via Σ_2 .

If we increase t and move in parameter space from (t_*, A_*) to $(t_* + \delta, A_*)$ then the triple zero z_1 of the discriminant splits into three zeros z_1, z_2, z_3 , where z_1 is real and z_2 and z_3 are non-real and each other complex conjugates. We take $\text{Im } z_2 > 0$. Then by continuity we find the sheet structure of the Riemann surface as shown in Figure 2.

Also by continuity the same sheet structure will hold throughout the region in the first quadrant of the (t, A) plane that is bounded by the three curves $A = A_j(t)$, $j = 1, 2, 3$, and $A > 0$. This region consists of the three pieces

- (1) $0 < t \leq t_*$, $0 < A < A_1(t)$,
- (2) $0 < t \leq t_*$, $A_2(t) < A < A_3(t)$,
- (3) $t_* < t < t_{**}$, $0 < A < A_3(t)$.

For each such (t, A) we have three branch points z_1, z_2, z_3 in sector S_0 with $z_1 > 0$, $\text{Im } z_2 > 0$ and $z_3 = \bar{z}_2$. There are six other branch points $\omega^j z_k$, $j = 1, 2$, $k = 1, 2, 3$, and we use the sheet structure of \mathcal{R} as in Figure 2.

3.2 Proof of Theorem 2.2 (a)

We take $t > t_*$ and $0 < A < A_3(t)$. From the previous section we then know the sheet structure of the Riemann surface. The surface has genus three, and we take the following

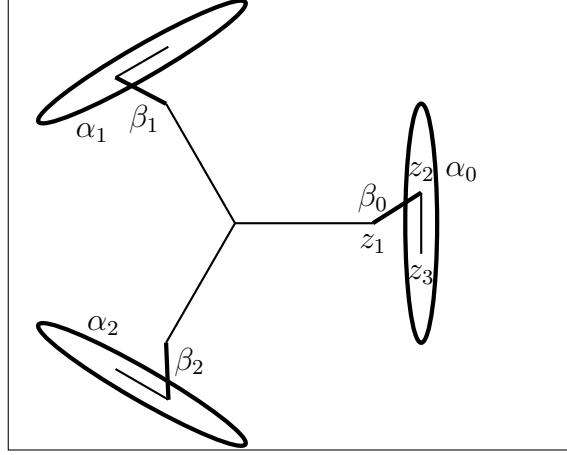


Figure 8: Cycles α_j , β_j , for $j = 0, 1, 2$ on the first sheet of the Riemann surface after deformation of the cuts. The β cycles also have a part on the second sheet. The α cycles are oriented counterclockwise and the cycle β_j is oriented from $\omega^j z_1$ to $\omega^j z_2$ on the first sheet

canonical homology basis

$$\{\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2\} \quad (3.5)$$

where α_0 is a nontrivial cycle going around $\Sigma_1^w \cap S_0$ on the first sheet with counterclockwise orientation, β_0 is a cycle from z_1 to z_2 on the first sheet, lying to the right of Σ_1^w , and back from z_2 to z_1 on the second sheet. The other cycles are obtained by rotation over angles $2\pi/3$ and $4\pi/3$, i.e.,

$$\alpha_j = \omega^j \alpha_0, \quad \beta_j = \omega^j \beta_0, \quad j = 1, 2.$$

To visualize these cycles it is convenient to first deform the whiskers Σ_1^w to cuts that connect the branch points $\omega^j z_2$ and $\omega^j z_3$ and are disjoint from $\omega^j z_1$, as in Figure 7. Then homotopic versions of the cycles (3.5) are shown in Figure 8.

Lemma 3.1. *ξdz has the Boutroux condition if and only if*

$$\operatorname{Re} \left(\oint_{\beta_0} \xi dz \right) = 0. \quad (3.6)$$

Proof. First observe that the residues of ξdz in the poles (the two points at infinity) are real; in fact they are $\pm t$, as can be deduced from (2.8).

Because of symmetry $\xi_1(\bar{z}) = \overline{\xi_1(z)}$ in the real axis, we have

$$\oint_{\alpha_0} \xi dz = \oint_{\alpha_0} \xi_1(z) dz \in i\mathbb{R}. \quad (3.7)$$

By the rotational \mathbb{Z}_3 symmetry $\xi_1(\omega z) = \omega^2 \xi_1(z)$, we then also have

$$\oint_{\alpha_j} \xi dz = \oint_{\alpha_0} \xi dz \in i\mathbb{R}, \quad j = 1, 2. \quad (3.8)$$

Similarly,

$$\oint_{\beta_j} \xi dz = \oint_{\beta_0} \xi dz, \quad j = 1, 2. \quad (3.9)$$

and therefore we have the Boutroux condition if and only if (3.6) is satisfied. \square

We define for $t \geq t_*$, $0 < A \leq A_3(t)$,

$$h(t, A) = \operatorname{Re} \left(\oint_{\beta_0} \xi dz \right) = \operatorname{Re} \left(\int_{z_1}^{z_2} (\xi_1(z) - \xi_2(z)) dz \right). \quad (3.10)$$

The second identity in (3.10) comes from the definition of β_0 as a path from z_1 to z_2 on the first sheet and back from z_2 to z_1 on the second sheet. Note that

$$h(t_*, A_1(t_*)) = 0, \quad (3.11)$$

since if we approach the critical values $t = t_*$, $A = A_1(t_*)$, the cycle β_0 shrinks to a point.

Lemma 3.2. *For each $t \in [t_*, 8]$ we have*

$$h(t, A_3(t)) > 0.$$

Proof. For $A = A_3(t)$ it is easy to calculate from (3.1) and (3.4) that the three zeros of Q are $w_1 = \frac{1}{12}(t-8)^2$ and $w_2 = w_3 = -3(1+t)$. Thus

$$\begin{aligned} z_1 &= 12^{-1/3}(8-t)^{2/3}, \\ z_2 &= 3^{1/3}(1+t)^{1/3}e^{\pi i/3}, \\ z_3 &= 3^{1/3}(1+t)^{1/3}e^{-\pi i/3} = \omega^2 z_2. \end{aligned} \quad (3.12)$$

Then z_2 , $\omega z_2 = -|z_2|$ and $\omega^2 z_2 = z_3$ are double branch points that connect all three sheets, and the three values ξ_1 , ξ_2 , ξ_3 coincide for these values of z . From the spectral curve equation (2.1) we have

$$\xi_1 + \xi_2 + \xi_3 = z^2$$

and it follows that

$$\xi_1(-|z_2|) = \xi_2(-|z_2|) = \xi_3(-|z_2|) = \frac{1}{3}|z_2|^2 = 3^{-1/3}(1+t)^{2/3}, \quad (3.13)$$

see also (3.12).

Now we calculate (3.10) by integrating $\xi_1 - \xi_2$ from z_1 to 0 and then from 0 to z_2 along the ray $\arg z = \pi/3$. The integral from z_1 to 0 does not contribute to the real part, since ξ_1 and ξ_2 are complex conjugates of each other there. What remains is

$$\begin{aligned} h(t, A_3(t)) &= \operatorname{Re} \left(\int_0^{z_2} (\xi_1(z) - \xi_{2,-}(z)) dz \right) \\ &= \operatorname{Re} \left(e^{\pi i/3} \int_0^{|z_2|} [\xi_1(re^{\pi i/3}) - \xi_{2,-}(re^{\pi i/3})] dr \right) \end{aligned} \quad (3.14)$$

where we put $z = re^{\pi i/3}$. The value $\xi_{2,-}$ in (3.14) denotes the limit of ξ_2 as we approach $\arg z = \pi/3$ from the sector S_0 . This is the limiting value from the right if we orient $\arg z = \pi/3$ from 0 to ∞ .

By the symmetry $\xi_j(z) = \omega \xi_j(\omega z)$, we have $e^{\pi i/3} \xi_1(re^{\pi i/3}) = -\xi_1(-r)$, $e^{\pi i/3} \xi_{2,-}(re^{\pi i/3}) = -\xi_{2,+}(-r)$, where $\xi_{2,+}$ denotes the limit from the upper half plane. Thus by putting $s = -r$ in (3.14) we obtain

$$h(t, A_3(t)) = \operatorname{Re} \left(\int_{-|z_2|}^0 (\xi_{2,+}(s) - \xi_1(s)) ds \right) \quad (3.15)$$

where the negative real axis is oriented from left to right. For $s < 0$ we have that $\xi_1(s)$ is real and $\xi_{2,+}(s)$ and $\xi_{3,+}(s)$ are complex conjugate. Since $\xi_1 + \xi_2 + \xi_3 = z^2$, we have $\operatorname{Re} \xi_{2,+}(s) = \frac{1}{2}(s^2 - \xi_1(s))$ for $s < 0$, and so by (3.15)

$$h(t, A_3(t)) = \frac{1}{2} \int_{-|z_2|}^0 (s^2 - 3\xi_1(s)) ds \quad (3.16)$$

Putting $z = s < 0$, $\xi = \frac{1}{3}s^2$, $A = A_3(t)$, in the spectral curve equation (2.1) we obtain

$$P(\frac{1}{3}s^2, s) = -\frac{1}{27}(2s^3 + 3t - 24)(s^3 + 3t + 3)$$

which, since $t \leq 8$, has exactly one zero for $s < 0$, namely

$$s = -3^{1/3}(1+t)^{1/3} = -|z_2|,$$

see (3.12). It follows that

$$\xi_1(s) = \frac{1}{3}s^2 \quad \text{if and only if } s = -|z_2|. \quad (3.17)$$

[The equality $\xi(s) = \frac{1}{3}s^2$ for $s = -|z_2|$ is also immediate from (3.12) and (3.13).] As $s \rightarrow 0-$ we have $\xi_1(s) \rightarrow -A^{1/3}$ (this follows from (2.1)) and so $\xi_1(s) < \frac{1}{3}s^2$ for s close to 0. From (3.17) we then get

$$\xi_1(s) < \frac{1}{3}s^2 \quad \text{for } s \in (-|z_2|, 0)$$

and then (3.16) tells us that $h(t, A_3(t)) > 0$, which proves the lemma. \square

Lemma 3.3. *For all $t \geq t_*$ and $0 < A < A_3(t)$ we have*

$$\frac{\partial h}{\partial A} > 0. \quad (3.18)$$

Proof. Note that by (2.1)

$$\frac{\partial \xi}{\partial A} = -\frac{\frac{\partial P}{\partial A}}{\frac{\partial P}{\partial \xi}} = -\frac{1}{3\xi^2 - 2z^2\xi - (1+t)z} \quad (3.19)$$

is a meromorphic function on the Riemann surface with simple poles at the branch points $\omega^j z_k$, $j, k = 1, 2, 3$, since these are the points where $\frac{\partial P}{\partial \xi} = 0$. The meromorphic differential dz has a zero at the branch points, and therefore $\frac{\partial \xi}{\partial A} dz$ is a meromorphic differential whose only possible poles are at the points at infinity.

Because of (2.8) and (3.19) we have

$$\begin{aligned} \frac{\partial \xi_1}{\partial A}(z) &= O(z^{-4}) & \text{as } z \rightarrow \infty, \\ \frac{\partial \xi_2}{\partial A}(z) &= O(z^{-5/2}) & \text{as } z \rightarrow \infty, z \in S_0, \end{aligned}$$

Then it easily follows that the singularities at the points at infinity are removable, and so $\frac{\partial \xi}{\partial A} dz$ is a holomorphic differential. It is a multiple of the holomorphic differential ω_R given in (2.27). The holomorphic differential has a double zero at both points at infinity, and these are the only zeros of $\frac{\partial \xi}{\partial A} dz$, since the genus is 3.

Because of the symmetries (3.8)–(3.9), we have for $j = 1, 2$,

$$\begin{aligned} \oint_{\alpha_j} \frac{\partial \xi}{\partial A} dz &= \oint_{\alpha_0} \frac{\partial \xi}{\partial A} dz \in i\mathbb{R}, \\ \oint_{\beta_j} \frac{\partial \xi}{\partial A} dz &= \oint_{\beta_0} \frac{\partial \xi}{\partial A} dz. \end{aligned} \quad (3.20)$$

Not all periods of a non-zero holomorphic differential can be purely imaginary, see e.g. [12, Proposition III.3.3]. Hence from (3.20) and (3.10) we conclude that

$$\frac{\partial h}{\partial A} = \operatorname{Re} \left(\oint_{\beta_0} \frac{\partial \xi}{\partial A} dz \right) \neq 0.$$

Then by continuity in the parameters t and A , we either have $\frac{\partial h}{\partial A} > 0$, or $\frac{\partial h}{\partial A} < 0$, for all t, A with $t \geq t_*$ and $0 < A < A_3(t)$. Since $h(t, A_3(t)) > 0$ by Lemma (3.2) and $h(t_*, A_1(t_*)) = 0$ by (3.11), we have the first possibility, and the lemma is proved. \square

After these preparations it is easy to prove part (a) of Theorem 2.2.

Proof of Theorem 2.2 (a). Since $A \mapsto h(t_*, A)$ is increasing because of Lemma 3.3, we have $h(t_*, 0) < h(t_*, A_1(t_*)) = 0$, see (3.11). By Lemma 3.2, we have $h(8, 0) > 0$ and so by continuity there is $t_{**} \in (t_*, 8)$ such that $h(t_{**}, 0) = 0$ and $h(t, 0) < 0$ for every $t \in [t_*, t_{**})$. Let $t \in (t_*, t_{**})$. Since $h(t, 0) < 0$ and $h(t, A_3(t)) > 0$ (see Lemma 3.3) there is a value $A = A(t) \in (0, A_3(t))$ such that $h(t, A(t)) = 0$. This value for A is unique because of Lemma 3.3, and so $t \mapsto A(t)$ is continuous with $A(t_{**}) = 0$. For this value of $A = A(t)$ we have the Boutroux condition by Lemma 3.1 and the definition (3.10) of $h(t, A)$. The fact that $h(t_{**}, 0) = 0$ implies (2.11). This proves part (a). \square

3.3 Proof of Theorem 2.2 (b)

Let $t \in (t_*, t_{**})$ and $A = A(t)$. We define

$$H(z) = \operatorname{Re} \int_{z_1}^z (\xi_1(s) - \xi_2(s)) ds, \quad z \in S_0 \setminus [0, z_1] \quad (3.21)$$

with a path of integration in $S_0 \setminus [0, z_1]$. Because of the Boutroux condition (2.6) $H(z)$ is well-defined, and it is independent of the path from z_1 to z . Indeed, if we take two paths γ_1 and γ_2 in S_0 with corresponding values H_1 and H_2 then

$$H_1(z) - H_2(z) = \operatorname{Re} \int_{\gamma_2^{-1} \circ \gamma_1} (\xi_1(s) - \xi_2(s)) ds$$

which can be identified as the real part of $\oint_{\gamma} \xi ds$ for a closed curve γ on the Riemann surface and the real part is zero because of (2.6).

Then H is a well-defined harmonic function on $S_0 \setminus [0, z_1]$ and it extends to a continuous function on \bar{S}_0 . Its level sets $H(z) = c$ are the trajectories of the quadratic differential $-(\xi_1 - \xi_2)^2 ds^2$, see [26]. Also by Lemma (3.1) and (3.10), (3.21) we have $H(z_2) = 0$. By the

local theory of quadratic differential near a simple zero [26], there are three trajectories emanating from z_2 that are on the zero level set $H(z) = 0$. Similarly, H is zero on three trajectories from z_3 .

Lemma 3.4. *H has exactly one zero on the half-ray $\arg z = \pi/3$.*

Proof. For $z = xe^{\pi i/3}$, $x > 0$, we integrate from z_1 to z by first going from z_1 to 0 and then from 0 to z along the half ray of angle $\pi/3$. Then as in the proof of Lemma 3.2 we find from (3.21) that

$$\begin{aligned} H(xe^{\pi i/3}) &= \operatorname{Re} \left(\int_0^z (\xi_1(s) - \xi_{2,-}(s)) ds \right) \\ &= \frac{1}{2} \int_{-x}^0 (s^2 - 3\xi_1(s)) ds, \end{aligned} \quad (3.22)$$

see, in particular, (3.16).

The function $s^2 - 3\xi_1(s)$ satisfies

$$\begin{aligned} s^2 - 3\xi_1(s) &= -2s^2 + O(s^{-1}), \quad \text{as } s \rightarrow -\infty, \\ s^2 - 3\xi_1(s) &\rightarrow 3A^{1/3} > 0 \quad \text{as } s \rightarrow 0-, \end{aligned} \quad (3.23)$$

. Hence it changes sign at least once on the negative real axis. Suppose s^* is a zero of $s^2 - 3\xi_1(s)$ on the negative real axis. Then, inserting $\xi_1(s^*) = \frac{1}{3}(s^*)^2$ into the cubic equation (2.1), we find that $w^* = (s^*)^3$ is a zero of

$$w^2 - \frac{9(2-t)}{2}w - \frac{27}{2}A.$$

This quadratic polynomial has one positive zero and one negative zero (since $A > 0$). Thus w^* is the unique negative root, and then s^* is unique as the negative real solution of $s^2 - 3\xi_1(s)$ and together with (3.23) we find

$$s^2 - 3\xi_1(s) \begin{cases} < 0 & \text{for } s < s^* \\ > 0 & \text{for } s^* < s < 0. \end{cases}$$

Then by (3.22) we find that $x \mapsto H(xe^{\pi i/3})$ is strictly increasing for $0 < x < -s^*$ and strictly decreasing for $x > -s^*$. Then there is exactly one zero on the half ray $\arg z = \pi/3$, since

$$\begin{aligned} H(xe^{\pi i/3}) &= -\frac{1}{3}x^3 + O(\log x), \quad \text{as } x \rightarrow +\infty, \\ H(xe^{\pi i/3}) &\rightarrow 0, \quad \text{as } x \rightarrow 0+, \end{aligned}$$

see (3.22) and (3.23). □

Lemma 3.5. *As $z \rightarrow \infty$ in S_0 ,*

$$H(z) = \operatorname{Re} \left(\frac{1}{3}z^3 - \frac{2}{3}z^{3/2} + O(\log |z|) \right).$$

Proof. This is immediate from (2.8) and (3.21). \square

From Lemma 3.5 it follows that there are two unbounded branches of the zero level set $H(z) = 0$ in S_0 . They tend to infinity at asymptotic angles $\pm\pi/6$, respectively.

Lemma 3.6. *For x real and $x > z_1$ we have $H(x) > 0$.*

Proof. There is no solution of $\xi_1(s) = \xi_2(s)$ for $s > z_1$, since such a solution would show up as a zero of the discriminant of P , and in the supercritical case the zeros z_2 and z_3 are not real. Since $\xi_{1,2}(s) \in \mathbb{R}$ when $s > z_1$ we conclude that $\xi_1(s) - \xi_2(s)$ has a constant sign for $s > z_1$. Since $\xi_1(s) - \xi_2(s) = s^2 + O(s^{1/2})$ as $s \rightarrow +\infty$, see (2.8), the sign is positive. Then $H(x) > 0$ for $x > z_1$ by the definition (3.21) of H . \square

With the help of these lemmas we can now prove part (b) of Theorem 2.2.

Proof of Theorem 2.2 (b). We already noted that the trajectories of the quadratic differential $-(\xi_1 - \xi_2)^2 ds^2$ that emanate from z_2 are contained in the level set $H(z) = 0$ of H . There are three such trajectories, and we follow them in the sector

$$S_0^+ = \{z \in \mathbb{C} \mid 0 < \arg z < \pi/3\}.$$

The trajectories cannot exit S_0^+ along $(z_1, +\infty)$ by Lemma 3.6. They cannot go to a point in $[0, z_1)$ either, since the interval $[0, z_1]$ is a trajectory as well, and trajectories do not intersect, except possibly at zeros of the quadratic differential. Thus, if one of the trajectories from z_2 comes to the real axis, it will come to z_1 .

There is at most one trajectory that comes to the ray $\arg z = \pi/3$, because of Lemma 3.4. Any unbounded trajectory that stays inside S_0 has to go to infinity with asymptotic angle $\pi/6$. This follows from Lemma 3.5. There can be at most one such trajectory from z_2 .

Combining all this we see that the only possible topology of the trajectories emanating from z_2 (that is, of zero level sets of $H(z)$) is: one of the trajectories from z_2 has to come to z_1 , the other one intersects the ray $\arg z = \frac{\pi}{3}$ and the third one goes to infinity, see Figure 9. The trajectory connecting z_1 and z_2 forms the analytic arc $\gamma_{1,2}$, which proves part (b) of Theorem 2.2. \square

Remark 3.7. In the proof of Theorem 2.2 (b) we, in fact, have shown that $H(z) > 0$ on both sides of $\Sigma_1 \cap S_0$.

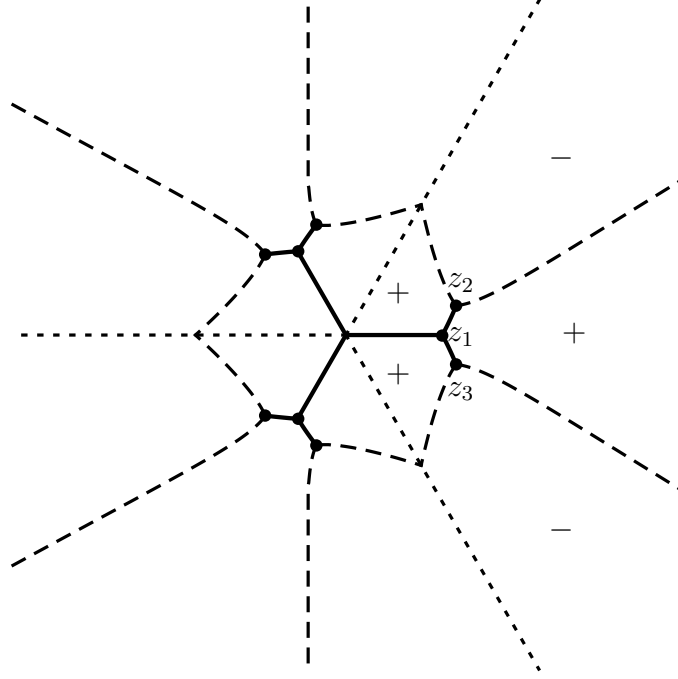


Figure 9: The zero level set of $H(z)$ given by (3.21) consists of the part of Σ_1 (solid lines) in S_0 , and two trajectories that emanate from each of the branch points z_2, z_3 (dashed lines). The figure also shows the sign of $H(z)$ in the sector S_0 . The set Σ_2 is shown with dotted lines. The figure is based on numerical calculations for the value $t = 0.155$.

3.4 Proof of Theorem 2.2 (c)

The measure μ_1 defined by (2.15) is real on the whiskers in S_0 because of the properties (2.12) and (2.13). It is also real on $[0, z_1]$, since $\xi_{2,+}(s)$ and $\xi_{1,+}(s)$ are each others complex conjugates for $s \in [0, z_1]$. Hence μ_1 is a real measure on $\Sigma_1 \cap S_0$. Because of \mathbb{Z}_3 symmetry it then also follows that μ_1 is real on the full Σ_1 .

Since $\xi_{2,+}(s)$ and $\xi_{1,+}(s)$ are only the same at the branch points, we then see that the density of μ_1 does not change sign on any arc $[0, z_1]$, $\gamma_{1,2}$ and $\gamma_{1,3}$, which implies that μ_1^o and μ_1^w are either positive or negative measures, where μ_1^o and μ_1^w denote the restriction of μ_1 to Σ_1^o and Σ_1^w , respectively. For $t = t^*$ we know from [5] that $\mu_1^o = \mu_1$ is positive and so by continuity in t , μ_1^o is positive for every $t \in (t_*, t_{**})$.

Now note that $\xi_1(x) > \xi_2(x)$ for $x \in (z_1, \infty)$ (we saw this in the proof of Lemma 3.6) and

$$\xi_1(s) - \xi_2(s) = c(s - z_1)^{1/2}(1 + O(s - z_1)) \quad \text{as } s \rightarrow z_1, s \in S_0 \quad (3.24)$$

for some positive constant $c > 0$. Then for $z \in \gamma_{1,2}$,

$$\begin{aligned} \int_{z_1}^z d\mu_1(s) &= \frac{1}{2\pi it} \int_{z_1}^z (\xi_{1,-}(s) - \xi_{2,-}(s)) ds \\ &= \frac{1}{2\pi it} \int_{z_1}^z c(s - z_1)^{1/2} ds (1 + O(z - z_1)) \\ &= \frac{1}{2\pi it} \frac{2c}{3} (z - z_1)^{3/2} (1 + O(z - z_1)) \quad \text{as } z \rightarrow z_1 \end{aligned}$$

which is positive for $z \in \gamma_{1,2}$ close enough to z_1 , since $\arg(z - z_1) \rightarrow \pi/3$ as $z \rightarrow z_1$. Thus μ_1 is not a negative measure on $\gamma_{1,2}$ and thus has to be positive on $\gamma_{1,2}$. Similarly it is positive on $\gamma_{1,3}$. Thus μ_1^w is a positive measure as well.

The total mass of μ_1 is $\frac{1}{2\pi it} \int_{\Sigma_1} (\xi_{1,-}(s) - \xi_{1,+}(s)) ds$ by (2.15), which can be written as a contour integral

$$\int d\mu_1 = \frac{1}{2\pi it} \oint_C \xi_1(s) ds,$$

where C is a contour that goes around Σ_1 in counterclockwise direction. By deforming the contour to infinity, and noting the expansion (2.3) with the residue t at infinity, we find that $\int d\mu_1 = 1$. Thus μ_1 is indeed a probability measure on Σ_1 .

3.5 Proof of Theorem 2.2 (d)

We start by studying the equation (2.16) and show that it is indeed the boundary of domain $\Omega(t)$.

Lemma 3.8. *Let $t > \frac{1}{8}$ and $A > 0$. Then for each $\theta \in [-\pi, \pi]$ there is a unique $r > 0$ such that $z = re^{i\theta}$ satisfies the equation (2.16).*

Proof. Putting $z = re^{i\theta}$ into the left hand-side (2.16) we find

$$2r^3 \cos(3\theta) - r^4 - (1+t)r^2 + A, \tag{3.25}$$

whose derivative with respect to r is

$$\begin{aligned} 6r^2 \cos(3\theta) - 4r^3 - 2(1+t)r &\leq 6r^2 - 4r^3 - 2(1+t)r \\ &= \frac{r}{4} [1 - 8t - (3 - 4r)^2] \end{aligned}$$

which is < 0 for all $r > 0$ since $t > \frac{1}{8}$. Thus (3.25) is strictly decreasing in r . The value for $r = 0$ is equal to $A > 0$, and it tends to $-\infty$ for $r \rightarrow \infty$. There is a unique $r > 0$ for which (3.25) is zero, which proves the lemma. \square

Lemma 3.8 shows that the curve given in polar coordinates by

$$2r^3 \cos(3\theta) - r^4 - (1+t)r^2 + A = 0$$

is the boundary of a starshaped domain that contains the origin. If $A = A(t)$ then it agrees with the equation (2.16) for $\partial\Omega(t)$, which arises from putting

$$\xi = \bar{z} \tag{3.26}$$

in the algebraic equation (2.1). For $t = t_*$ and $A = A_1(t_*)$ we know from [5] that $\partial\Omega(t)$ is given by $\xi_1(z) = \bar{z}$. That is, if $z \in \partial\Omega(t)$, then the relevant solution of (2.1) which gives rise to (3.26) is the solution $\xi_1(z)$, which is defined on the first sheet of the Riemann surface.

Lemma 3.9. *For each $t \in (t_*, t_{**})$ we have $z_1 \in \Omega(t)$ and $z_2, z_3 \in \mathbb{C} \setminus \Omega(t)$.*

Proof. Since z_1 is a branch point of (2.1),

$$\frac{\partial P}{\partial \xi} = 3\xi^2 - 2z^2\xi - (1+t)z = 0$$

for $z = z_1$ and $\xi = \xi_1(z_1) = \xi_2(z_1)$. Note that $\xi_2(x) > 0$ for large positive x due to (2.8), and ξ_2 does not have any zeros on $[z_1, \infty)$ as can be easily checked from the algebraic equation (2.1). Thus $\xi_2(z_1) > 0$. Solving the quadratic equation, we then get $\xi_2(z_1) = F(z_1)$ where

$$F(x) = \frac{1}{3} \left[x^2 + \sqrt{x^4 + 3(1+t)x} \right].$$

It is an easy calculus exercise to show that for $t > \frac{1}{8}$ we have

$$F(x) > \frac{1}{3} \left[x^2 + \sqrt{x^4 + \frac{27}{8}x} \right] \geq x, \quad x > 0.$$

Thus

$$\xi_2(z_1) > z_1.$$

Now note that $\xi_2(x)$ is real for real $x > z_1$ and it behaves like $x^{1/2}$ as $x \rightarrow \infty$ by (2.8), so that clearly $\xi_2(x) < x$ for large enough $x > z_1$. Thus by continuity there exists $\tilde{x} > z_1$ such that $\xi_2(\tilde{x}) = \tilde{x}$. Then \tilde{x} belongs to $\partial\Omega(t)$ since (3.26) is satisfied, and noting Lemma 3.8, we conclude that it is the unique intersection point of $\partial\Omega(t)$ with the positive real axis. Since $\tilde{x} > z_1$ we find that z_1 lies in $\Omega(t)$.

We subdivide $\partial\Omega$ into pieces

$$(\partial\Omega)_j = \{z \in \mathbb{C} \mid \xi_j(z) = \bar{z}\}, \quad j = 1, 2, 3,$$

where from now on we drop the t -dependence from the notation. We visualize $\partial\Omega$ on the Riemann surface, by putting the part $(\partial\Omega)_j$ on sheet j . For $t \leq 1/8$ we have $\partial\Omega = (\partial\Omega)_1$, and so $\partial\Omega$ is fully on the first sheet. For $t = 1/8$, the curve contains the branch points $\omega^j z_1^*$, $j = 1, 2, 3$, and the structure of Σ_1 changes near these branch points if we move into the supercritical regime $t > 1/8$. Then part of $\partial\Omega$ may move to the second sheet, and this happens indeed since we just proved that $\tilde{x} \in \partial\Omega$ with $\xi_2(\tilde{x}) = \tilde{x}$. So the part of $\partial\Omega$ near the real axis is on the second sheet, as well as the parts near the halfrays at angles $\pm 2\pi/3$. The remaining parts are on the first sheet, and in particular the parts near the angles $\pm\pi/3$ and π , where the branch cut from the second to the third sheet is. The curve can then never move to the third sheet, and it follows that $(\partial\Omega)_3 = \emptyset$ for all $t \in (t_*, t_{**})$. The curve intersects the whiskers $\gamma_{1,2}$ and $\gamma_{1,3}$. Each intersection will mean a change from $(\partial\Omega)_1$ to $(\partial\Omega)_2$ or vice versa. So there are an odd number of intersections¹, and this means that z_2 is outside of Ω . By symmetry with respect to complex conjugation, also $z_3 \notin \Omega(t)$ and the lemma is proved. \square

We are now ready for the proof of part (d).

Proof of Theorem 2.2 (d). We give the proof under the assumption that $\partial\Omega$ has one intersection point with $\gamma_{1,2}$. The proof can be modified to cover the hypothetical situation of more than one intersection point (which probably does not occur).

Since $\xi_1(z) = z^2 + tz^{-1} + O(z^{-4})$ as $z \rightarrow \infty$, we have for a nonnegative integer k ,

$$t_k := \frac{1}{2\pi i} \oint_C \frac{\xi_1(s)}{s^k} ds = \begin{cases} t & \text{if } k = 0, \\ 1 & \text{if } k = 3, \\ 0 & \text{otherwise,} \end{cases}$$

where C is a contour that encircles Σ_1 once in counterclockwise direction.

We deform C inwards so that it consists of $\partial\Omega$ and the plus and minus sides of the parts of Σ_1^w that are outside of Ω . We use $(\partial\Omega)_1$ and $(\partial\Omega)_2$ as in the proof of the last lemma. Then we have

$$\begin{aligned} t_k &= \frac{1}{2\pi i} \int_{(\partial\Omega)_1 \cup (\partial\Omega)_2} \frac{\xi_1(s)}{s^k} ds + \frac{1}{2\pi i} \int_{\Sigma_1^w \setminus \Omega} \frac{(\xi_{1,-} - \xi_{1,+})(s)}{s^k} ds \\ &= \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\bar{s}}{s^k} ds + \frac{1}{2\pi i} \int_{(\partial\Omega)_2} \frac{\xi_1(s) - \xi_2(s)}{s^k} ds + \frac{1}{2\pi i} \int_{\Sigma_1^w \setminus \Omega} \frac{(\xi_{2,+} - \xi_{1,+})(s)}{s^k} ds. \end{aligned} \quad (3.27)$$

¹There is probably only one intersection, but we have not been able to prove this.

The integral over $(\partial\Omega)_2$ can be deformed to an integral over $\Sigma_1^w \cap \Omega$, which results in

$$\frac{1}{2\pi i} \int_{(\partial\Omega)_2} \frac{\xi_1(s) - \xi_2(s)}{s^k} ds = \frac{1}{2\pi i} \int_{\Sigma_1^w \cap \Omega} \frac{\xi_{2,+}(s) - \xi_{1,+}(s)}{s^k} ds. \quad (3.28)$$

From (3.27)–(3.28) we get

$$t_k = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\bar{s}}{s^k} ds + \frac{1}{2\pi i} \int_{\Sigma_1^w} \frac{(\xi_{2,+} - \xi_{1,+})(s)}{s^k} ds,$$

which gives (2.17) in view of (2.15). \square

The proof of Theorem 2.2 is now complete.

4 Proof of Theorem 2.4

4.1 The Riemann-Hilbert problem in the supercritical case

The proof of Theorem 2.4 is based on a steepest descent analysis of a RH problem for the orthogonal polynomials $P_{n,n}$ that we take from [5]. The RH problem arises from the fact that $P_{n,n}$ can be viewed as a multiple orthogonal polynomial with respect to two weight functions $w_{0,n}$ and $w_{1,n}$ on $\bigcup_{j=1}^3 \Gamma_j$, see Lemma 5.1 of [5].

We are free to move the contours in the complex plane, as long as we respect the starting and ending directions at infinity. We can then move them to a contour

$$\Gamma_Y = \Sigma_1 \cup \bigcup_{j=0}^2 C_j^\pm$$

where C_j^+ denotes the continuation of Σ_1 from $\omega^j z_2$ and C_j^- is the continuation of Σ_1 from $\omega^j z_3$, for $j = 0, 1, 2$, see Figure 10. Then the weights are expressed in terms of the Airy function Ai and its derivative (see Definition 5.3 of [5]). They are given on the parts of Γ_Y in the sector S_0 by

$$\begin{cases} w_{0,n}(x) = \text{Ai}\left(\frac{n^{2/3}}{t^{2/3}}x\right)e^{\frac{n}{3t}x^3} \\ w_{1,n}(x) = \text{Ai}'\left(\frac{n^{2/3}}{t^{2/3}}x\right)e^{\frac{n}{3t}x^3} \end{cases} \quad \text{on } [0, z_1],$$

$$\begin{cases} w_{0,n}(z) = \frac{1}{3}(\text{Ai}\left(\frac{n^{2/3}}{t^{2/3}}z\right) - \omega \text{Ai}\left(\omega \frac{n^{2/3}}{t^{2/3}}z\right))e^{\frac{n}{3t}z^3} \\ w_{1,n}(z) = \frac{1}{3}(\text{Ai}'\left(\frac{n^{2/3}}{t^{2/3}}z - \omega^2 \text{Ai}'\left(\omega \frac{n^{2/3}}{t^{2/3}}z\right)\right)e^{\frac{n}{3t}z^3} \end{cases} \quad \text{on } \gamma_{1,2} \cup C_0^+,$$

$$\begin{cases} w_{0,n}(x) = \frac{1}{3}(\text{Ai}\left(\frac{n^{2/3}}{t^{2/3}}z\right) - \omega^2 \text{Ai}\left(\omega^2 \frac{n^{2/3}}{t^{2/3}}z\right))e^{\frac{n}{3t}z^3} \\ w_{1,n}(z) = \frac{1}{3}(\text{Ai}'\left(\frac{n^{2/3}}{t^{2/3}}z - \omega \text{Ai}'\left(\omega^2 \frac{n^{2/3}}{t^{2/3}}z\right)\right)e^{\frac{n}{3t}z^3} \end{cases} \quad \text{on } \gamma_{1,3} \cup C_0^-,$$

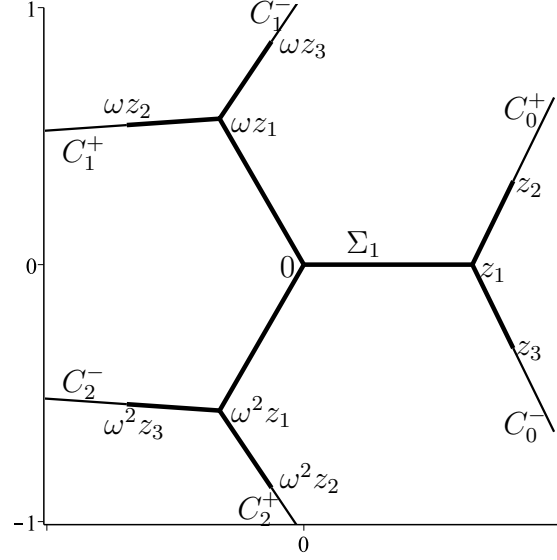


Figure 10: Contour Γ_Y in the RH problem for Y .

and then continued to other sectors by the symmetry property

$$w_{0,n}(\omega z) = \omega^2 w_{0,n}(z), \quad w_{1,n}(\omega z) = \omega w_{1,n}(z), \quad \text{for } z \in \Gamma_Y.$$

The RH problem then is:

RH problem 4.1. The matrix-valued function $Y : \mathbb{C} \setminus \Gamma_Y \rightarrow \mathbb{C}^{3 \times 3}$ satisfies

- Y is analytic.
- $Y_+(z) = Y_-(z) \begin{pmatrix} 1 & w_{0,n}(z) & w_{1,n}(z) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for $z \in \Gamma_Y$.
- $Y(z) = (I + O(1/z)) \text{diag}(z^n, z^{-\lceil n/2 \rceil}, z^{-\lfloor n/2 \rfloor})$ as $z \rightarrow \infty$.

The RH problem 4.1 has a unique solution if and only if $P_{n,n}$ uniquely exist and in that case

$$Y_{11}(z) = P_{n,n}(z). \quad (4.1)$$

In what follows we assume that n is even, since it will simplify the exposition. With minor modifications, the analysis can be done in the case of odd n as well.

The asymptotic analysis of the RH problem 4.1 will be obtained through a chain of transformations. In the end, we will obtain a “model” RH problem 4.6 for a matrix $M(z)$ (also known as a global parametrix), that gives the leading pre-exponential behavior of $Y(z)$ as $n \rightarrow \infty$, and a number of local parametrices (RH problem 4.8), that are used to obtain the error estimate. This chain of transformations is fully described in [5] for the subcritical case $t < t_*$ and constitutes a substantial part of that paper. In order to limit the size of the present paper, we will not repeat the full description of transformations from [5] here, but rather underline the distinctions of the supercritical case, caused by the presence of the whiskers.

The transformations $Y \mapsto X \mapsto V$ are the same as in [5]. The precise form of these transformations is irrelevant for the present paper. We only need to know the resulting RH problem, and the fact that the 11-entry remains unchanged, so that by (4.1),

$$V_{11}(z) = Y_{11}(z) = P_{n,n}(z). \quad (4.2)$$

To state the RH problem for V we introduce

$$Q_1(z) = \begin{cases} \frac{2}{3t}z^{3/2} - \frac{1}{3t}z^3, & z \in S_0, \\ -\frac{2}{3t}z^{3/2} - \frac{1}{3t}z^3, & z \in S_1 \cup S_2, \end{cases} \quad (4.3)$$

$$Q_2(z) = \begin{cases} \frac{4}{3t}z^{3/2}, & 0 < \arg z < 2\pi/3 \text{ or } -\pi < \arg z < -2\pi/3, \\ -\frac{4}{3t}z^{3/2}, & -2\pi/3 < \arg z < 0 \text{ or } 2\pi/3 < \arg z < \pi, \end{cases} \quad (4.4)$$

with principal branches of the fractional powers. The definitions are such that $Q_j(\omega z) = Q_j(z)$ for $j = 1, 2$. We also define the constant

$$\alpha = \frac{1}{2} + \frac{i}{6}\sqrt{3} = \frac{\sqrt{3}}{3}e^{\pi i/6}. \quad (4.5)$$

Then the RH problem for V is, see [5, RH problem 6.7]:

RH problem 4.2. $V : \mathbb{C} \setminus \Gamma_V \rightarrow \mathbb{C}^{3 \times 3}$, where $\Gamma_V = \Gamma_Y \cup \Sigma_2$ and Σ_2 is given by (2.10), satisfies

- V is analytic

- $V_+ = V_- J_V$ on Γ_V with

$$J_V(z) = \begin{cases} \begin{pmatrix} 1 & e^{-nQ_1(z)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Sigma_1^o = \bigcup_{j=0}^2 [0, \omega^j z_1] \\ \begin{pmatrix} 1 & \alpha e^{-nQ_1(z)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \bigcup_{j=0}^2 \omega^j (\gamma_{1,2} \cup C_0^+) \\ \begin{pmatrix} 1 & \bar{\alpha} e^{-nQ_1(z)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \bigcup_{j=0}^2 \omega^j (\gamma_{1,3} \cup C_0^-) \\ \begin{pmatrix} 1 & 0 & \bar{\alpha} e^{-nQ_{1,-}(z)} \\ 0 & \omega^2 e^{nQ_2(z)} & 1 \\ 0 & 0 & \omega e^{-nQ_2(z)} \end{pmatrix}, & z \in \Sigma_2. \end{cases} \quad (4.6)$$

- $V(z) = (I + O(1/z))A(z) \begin{pmatrix} z^n & 0 & 0 \\ 0 & z^{-n/2} & 0 \\ 0 & 0 & z^{-n/2} \end{pmatrix}$ as $z \rightarrow \infty$, where $A(z)$ is given by,

$$A(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^{1/4} & 0 \\ 0 & 0 & z^{-1/4} \end{pmatrix} \times \frac{1}{\sqrt{2}} \begin{cases} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & -i \\ 0 & -i & 1 \end{pmatrix}, & z \in S_0, \\ \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & i & 1 \\ 0 & -1 & i \end{pmatrix}, & z \in S_1, \\ \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & -i & -1 \\ 0 & 1 & i \end{pmatrix}, & z \in S_2, \end{cases} \quad (4.7)$$

see also formula (6.9) in [5].

The next transformation $V \mapsto U$ uses the measures μ_1 and μ_2 , see (2.22), and their g functions

$$g_j(z) = \int \log(z - s) d\mu_j(s), \quad j = 1, 2. \quad (4.8)$$

The branches for the logarithm are chosen as in [5, Section 6.3]. It means that g_1 is defined and analytic in $\mathbb{C} \setminus (\Sigma_1 \cup \mathbb{R}^-)$, g_2 is analytic in $\mathbb{C} \setminus \Sigma_2$ with symmetries

$$g_1(\omega^\pm z) = g_1(z) \pm 2\pi i/3, \quad g_2(\omega^\pm z) = g_2(z) \pm \pi i/3, \quad \text{for } z \in S_0.$$

In addition $g_1(x)$ and $g_2(x)$ are real for real $x > z_1$, see [5, Section 6.3].

Then, as in [5, Lemma 6.8], there is a constant ℓ such that

$$g_{1,+}(z) + g_{1,-}(z) - g_2(z) = \frac{2}{3t}z^{3/2} - \frac{1}{3t}z^3 + \ell, \quad z \in [0, z_1].$$

On the whiskers, we then have

$$g_{1,+}(z) + g_{1,-}(z) - g_2(z) = \frac{2}{3t}z^{3/2} - \frac{1}{3t}z^3 + \ell \begin{cases} +2\pi i\beta, & z \in \gamma_{1,2}, \\ -2\pi i\beta, & z \in \gamma_{1,3}, \end{cases} \quad (4.9)$$

where $\beta = \mu_1(\gamma_{1,2}) = \mu_1(\gamma_{1,3})$, which by the symmetry is the same as $\beta = \frac{1}{6}\mu_1(\Sigma_1^w)$, see (2.32).

The definition of U is as in Definition 6.9 of [5], namely

$$U(z) = \begin{pmatrix} e^{-n\ell} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} V(z) \begin{pmatrix} e^{-n(g_1(z)-\ell)} & 0 & 0 \\ 0 & e^{n(g_1(z)-g_2(z))} & 0 \\ 0 & 0 & e^{ng_2(z)} \end{pmatrix}. \quad (4.10)$$

It then follows that

$$U_{11}(z) = P_{n,n}(z)e^{-ng_1(z)}, \quad z \in \mathbb{C} \setminus \Gamma_U, \quad (4.11)$$

with $\Gamma_U = \Gamma_V = \Gamma_Y \cup \Sigma_2$.

The jumps in the RH problem for U are conveniently expressed in terms of the functions

$$\varphi_1(z) = \frac{1}{2t} \int_{\omega^j z_1}^z (\xi_1(s) - \xi_2(s)) ds \quad \text{for } z \in S_j \setminus \Sigma_1, \quad j = 0, 1, 2, \quad (4.12)$$

$$\varphi_2(z) = \frac{1}{2t} \int_0^z (\xi_2(s) - \xi_3(s)) ds \mp \frac{\pi i}{6} \quad \text{for } z \in S_0 \cap \mathbb{C}^\pm \setminus \Sigma_1, \quad (4.13)$$

with formulas similar to (4.13) in $S_1 \setminus \Sigma_1$ and $S_2 \setminus \Sigma_1$ as in [5, formula (6.23)]. The path of integration in (4.12) goes from $\omega^j z_1$ to $\omega^j(z_1 + \varepsilon)$ for some $\varepsilon > 0$ and then continues to z in the domain $S_j \setminus \Sigma_1$. The path of integration in (4.13) is in $S_0^\pm \setminus \Sigma_1$, where $S_0^\pm = S_0 \cap \mathbb{C}^\pm$. Note that

$$\operatorname{Re} \varphi_1(z) = \frac{1}{2t} H(z) \quad (4.14)$$

where H was given by (3.21).

We remind, see (2.32), that

$$\beta = \frac{1}{6}\mu_1(\Sigma_1^w) = \mu_1(\gamma_{1,2}) \in (0, \frac{1}{6}). \quad (4.15)$$

This number appears in the RH problem for U . The reason for this is that it appears in the jump relation (4.9), and it also comes in the jump of φ_1 on Σ_1 . Indeed, from the definition of φ_1 (in particular the choice of the path from $\omega^j z_1$ to z) and (2.15), (4.15), one finds

$$\varphi_{1,+}(z) + \varphi_{1,-}(z) = \begin{cases} 0 & \text{on } \Sigma_1^o, \\ 2\pi i\beta & \text{on } \bigcup_{j=0}^2 \omega^j \gamma_{1,2}, \\ -2\pi i\beta & \text{on } \bigcup_{j=0}^2 \omega^j \gamma_{1,3}. \end{cases} \quad (4.16)$$

We will not give the details for the (sometimes tedious) calculations that produce a list of identities that lead to the following RH problem.

RH problem 4.3. U is the solution of the following RH problem.

- U is analytic in $\mathbb{C} \setminus \Gamma_U$, where $\Gamma_U = \Gamma_V$.
- $U_+ = U_- J_U$ on Γ_U with

$$J_U(z) = \begin{cases} \begin{pmatrix} e^{-2n\varphi_{1,+}(z)} & 1 & 0 \\ 0 & e^{-2n\varphi_{1,-}(z)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Sigma_1^o, \\ \begin{pmatrix} e^{-2n\varphi_{1,+}(z)+2\pi i n\beta} & \alpha e^{2\pi i n\beta} & 0 \\ 0 & e^{-2n\varphi_{1,-}(z)+2\pi i n\beta} & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \bigcup_{j=0}^2 \omega^j \gamma_{1,2}, \\ \begin{pmatrix} e^{-2n\varphi_{1,+}(z)-2\pi i n\beta} & \bar{\alpha} e^{-2\pi i n\beta} & 0 \\ 0 & e^{-2n\varphi_{1,-}(z)-2\pi i n\beta} & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \bigcup_{j=0}^2 \omega^j \gamma_{1,3}, \\ \begin{pmatrix} 1 & 0 & \bar{\alpha} e^{2n\varphi_{1,-}(z)} \\ 0 & \omega^2 e^{-2n\varphi_{2,+}(z)} & 1 \\ 0 & 0 & \omega e^{-2n\varphi_{2,-}(z)} \end{pmatrix}, & z \in \Sigma_2, \end{cases} \quad (4.17)$$

and

$$J_U(z) = \begin{cases} \begin{pmatrix} 1 & \alpha e^{2n\varphi_1(z)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \bigcup_{j=0}^2 C_j^+ \\ \begin{pmatrix} 1 & \bar{\alpha} e^{2n\varphi_1(z)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \bigcup_{j=0}^2 C_j^-. \end{cases} \quad (4.18)$$

- $U(z) = (I + O(1/z))A(z)$ as $z \rightarrow \infty$, where $A(z)$ is given by (4.7).

Further transformations $U \mapsto T \mapsto S$ are again as in [5]. The transformation $U \mapsto T$ is effective in the domain bounded by C_0 , C_1 , C_2 and the whiskers. It simplifies the jumps, as it kills for example the 13-entry in the jump matrix on Σ_2 . The number α also disappears from the jumps, and only its real part $\operatorname{Re} \alpha = \frac{1}{2}$, see (4.5), remains in the jump matrices. It leads to the following RH problem.

RH problem 4.4. T is the solution of the following RH problem.

- T is analytic in $\mathbb{C} \setminus \Gamma_T$, where $\Gamma_T = \Gamma_U$.
- $T_+ = T_- J_T$ on Γ_T with

$$J_T(z) = \begin{cases} \begin{pmatrix} e^{-2n\varphi_{1,+}(z)} & 1 & 0 \\ 0 & e^{-2n\varphi_{1,-}(z)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Sigma_1^o, \\ \begin{pmatrix} e^{-2n\varphi_{1,+}(z)+2\pi i n\beta} & \frac{1}{2}e^{2\pi i n\beta} & 0 \\ 0 & e^{-2n\varphi_{1,-}(z)+2\pi i n\beta} & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \bigcup_{j=0}^2 \omega^j \gamma_{1,2}, \\ \begin{pmatrix} e^{-2n\varphi_{1,+}(z)-2\pi i n\beta} & \frac{1}{2}e^{-2\pi i n\beta} & 0 \\ 0 & e^{-2n\varphi_{1,-}(z)-2\pi i n\beta} & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \bigcup_{j=0}^2 \omega^j \gamma_{1,3}, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 e^{-2n\varphi_{2,+}(z)} & 1 \\ 0 & 0 & \omega e^{-2n\varphi_{2,-}(z)} \end{pmatrix}, & z \in \Sigma_2, \end{cases} \quad (4.19)$$

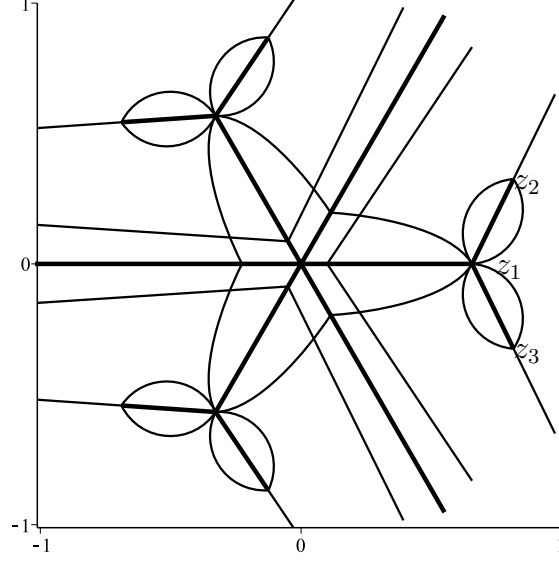


Figure 11: Contours Γ_S in the RH problem for S .

and

$$J_T(z) = \begin{cases} \begin{pmatrix} 1 & \frac{1}{2}e^{2n\varphi_1(z)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \bigcup_{j=0}^2 C_j^+ \\ \begin{pmatrix} 1 & \frac{1}{2}e^{2n\varphi_1(z)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \bigcup_{j=0}^2 C_j^-. \end{cases} \quad (4.20)$$

- $T(z) = (I + O(1/z))A(z)$ as $z \rightarrow \infty$, where $A(z)$ is given by (4.7).

In the step $T \mapsto S$ lenses are opened around Σ_1 and Σ_2 . It is based on a factorization of the jump matrices on these contours. This is the same as in [5], except that we now also have the whiskers in Σ_1 . The jump matrix J_T on $\bigcup_j \omega^j \gamma_{1,2}$, see (4.19), has the following factorization (we only list the nontrivial 2×2 block)

$$\begin{pmatrix} e^{-2n\varphi_{1,+}(z)+2\pi i n\beta} & \frac{1}{2}e^{2\pi i n\beta} \\ 0 & e^{-2n\varphi_{1,-}(z)+2\pi i n\beta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2e^{-2n\varphi_{1,-}(z)} & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2}e^{2\pi i n\beta} \\ 2e^{-2\pi i n\beta} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2e^{-2n\varphi_{1,+}(z)} & 1 \end{pmatrix} \quad (4.21)$$

and there is a similar factorization for the jump matrix J_T on $\bigcup_j \omega^j \gamma_{1,3}$.

The definition of S (which we do not specify here in detail) leads to a RH problem for S on a complicated set of contours, see Figure 11.

RH problem 4.5. S is the solution of the following RH problem.

- S is analytic in $\mathbb{C} \setminus \Gamma_S$, where Γ_S consists of Γ_U and lenses around Σ_1 and Σ_2 .
- $S_+ = S_- J_S$ on Γ_S with

$$J_S = \begin{cases} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{on } \Sigma_1^o, \\ \begin{pmatrix} 0 & \frac{1}{2}e^{2\pi i n \beta} & 0 \\ -2e^{-2\pi i n \beta} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{on } \bigcup_{j=0}^2 \omega^j \gamma_{1,2} \\ \begin{pmatrix} 0 & \frac{1}{2}e^{-2\pi i n \beta} & 0 \\ -2e^{2\pi i n \beta} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{on } \bigcup_{j=0}^2 \omega^j \gamma_{1,3} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} & \text{on } \Sigma_2, \end{cases} \quad (4.22)$$

and

$$J_S = \begin{pmatrix} 1 & \frac{1}{2}e^{2n\varphi_1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } \bigcup_{j=0}^2 C_j^\pm \quad (4.23)$$

on Γ_S , and on the lips of the lenses it is

$$J_S = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ e^{-2n\varphi_1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{array}{l} \text{on lips around } \Sigma_1^o \\ \text{outside lens around } \Sigma_2 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 2e^{-2n\varphi_1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{on lips around } \Sigma_1^w \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \omega^\mp e^{-2n\varphi_2} & 1 \end{pmatrix} & \begin{array}{l} \text{on lips around } \Sigma_2 \\ \text{outside lens around } \Sigma_1 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ e^{-2n\varphi_1} & 1 & 0 \\ \pm e^{-2n(\varphi_1+\varphi_2)} & 0 & 1 \end{pmatrix} & \begin{array}{l} \text{on lips around } \Sigma_1 \\ \text{inside lens around } \Sigma_1 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \omega^\pm e^{-2n(\varphi_1+\varphi_2)} & \omega^\mp e^{-2n\varphi_2} & 1 \end{pmatrix} & \begin{array}{l} \text{on lips around } \Sigma_2 \\ \text{inside lens around } \Sigma_1. \end{array} \end{cases} \quad (4.24)$$

- $S(z) = (I + O(1/z))A(z)$ as $z \rightarrow \infty$, where $A(z)$ is given by (4.7).

Notice that by (4.11) and the transformations $U \mapsto T \mapsto S$,

$$S_{11}(z) = T_{11}(z) = U_{11}(z) = P_{n,n}(z)e^{-ng_1(z)} \quad \text{for } z \text{ outside the lenses around } \Sigma_1. \quad (4.25)$$

As it was shown in [5], the jumps on the lips of the lenses are such that $J_S = I + O(e^{-cn})$ as $n \rightarrow \infty$ uniformly if we stay away from the branch points $\omega^j z_k$, $j = 0, 1, 2$, $k = 1, 2, 3$. Even though the lenses around whiskers were not considered in [5], the above estimate is valid on these lenses due to (4.14) and Remark 3.7. Ignoring the jumps that are close to the identity matrix, we arrive at a model RH problem.

4.2 The model problem (outer parametrix)

We ignore the jumps (4.24) on the lenses, as well as the jump (4.23) on $\bigcup_j C_j^\pm$, and we find the following model RH problem for a matrix $M = M_n$.

RH problem 4.6. • M is analytic in $\mathbb{C} \setminus (\Sigma_1 \cup \Sigma_2)$,

- $M_+ = M_- J_M$ on $\Sigma_1 \cup \Sigma_2$ with

$$J_M = \begin{cases} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{on } \Sigma_1^o \\ \begin{pmatrix} 0 & \frac{1}{2}e^{2\pi i n \beta} & 0 \\ -2e^{-2\pi i n \beta} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{on } \bigcup_{j=0}^2 \omega^j \gamma_{1,2} \\ \begin{pmatrix} 0 & \frac{1}{2}e^{-2\pi i n \beta} & 0 \\ -2e^{2\pi i n \beta} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{on } \bigcup_{j=0}^2 \omega^j \gamma_{1,3} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} & \text{on } \Sigma_2 \end{cases} \quad (4.26)$$

- $M(z) = (I + O(1/z))A(z)$ as $z \rightarrow \infty$ where $A(z)$ is given by (4.7)
- For $j = 0, 1, 2$ and $k = 1, 2, 3$ we have

$$M(z) = O\left((z - \omega^j z_k)^{-1/4}\right) \quad \text{as } z \rightarrow \omega^j z_k.$$

The model problem depends on $n\beta \bmod \mathbb{Z}$ with $n \in \mathbb{N}$ and $0 < \beta \frac{1}{6}$. The asymptotic condition (4.7) is compatible with the jump of M on the unbounded contour Σ_2 , since

$$A_+ = A_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \text{on } \Sigma_2.$$

as can be checked from the definition (4.7).

We write

$$\beta^* = \frac{1}{2} + \frac{\tau}{2\pi i} \log 2 + \int_{\infty_1}^{-A^{1/3}} \omega_R \quad (4.27)$$

see (2.27), (2.28) and (2.33) for the definitions of ω_R , τ , and $-A^{1/3}$.

Proposition 4.7. *The following holds:*

(a) *The model RH problem is solvable if and only if*

$$n\beta \neq \beta^* \pmod{\mathbb{Z}} \quad (4.28)$$

and if (4.28) holds then there is a unique solution $M = M_n$.

(b) *Let $\varepsilon > 0$ and $r > 0$, and let $\|\cdot\|$ denote any matrix norm. Then there is a constant $K > 0$ such that*

$$\|M_n^{-1}(z)\| \leq K, \quad \|M_n(z)\| \leq K$$

holds for all z such that $|z - \omega^j z_k| \geq r$ for all $j = 0, 1, 2$, $k = 1, 2, 3$, and all $n \in \mathbb{N}$ such that $\text{dist}(n\beta - \beta^, \mathbb{Z}) \geq \varepsilon$.*

(c) *The 11-entry of M_n has the form given in (2.36).*

The proof of Proposition 4.7 is rather long. We decided to put it in a separate section 5.

4.3 The local parametrix

Let $r > 0$ be a small number such that the disks $D(\omega^j z_k, r)$, $j = 0, 1, 2$, $k = 1, 2, 3$, are all disjoint and they do not intersect with the lens around Σ_2 . Let D denote the union of these nine disks.

The local parametrix P is defined in each of these disks and it should satisfy

RH problem 4.8. P satisfies the following:

- P is analytic in $D \setminus \Gamma_S$,
- $P_+ = P_- J_P$ on $\Gamma_S \cap D$ with $J_P = J_S$ is given by (4.22).
- $P(z) = (I + O(1/n))M_n(z)$ as $n \rightarrow \infty$ uniformly for z on the boundary of each of the disks.

We will be able to construct P with the desired jumps, but the required matching can be done only if n is restricted to

$$\mathbb{N}_\varepsilon = \{n \in \mathbb{N} \mid \text{dist}(n\beta - \beta^*, \mathbb{Z}) \geq \varepsilon\}$$

for some $\varepsilon > 0$. Note that this agrees with the earlier definition (2.33), in view of (4.27).

The construction can be done in each of the disks with Airy functions. We will do it in some detail for the point z_1 because of the somewhat unusual fact that three pieces of Σ_1 are connected at z_1 and, as a result of the opening of lenses, that nine curves from Γ_S

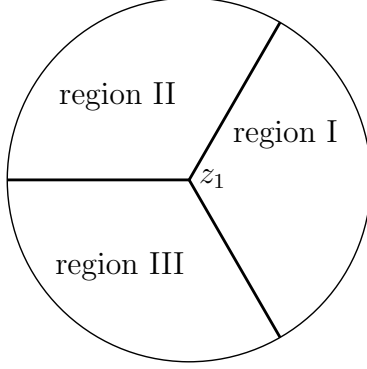


Figure 12: Disk $D(z_1, r)$ around z_1 and the parts of Σ_1 inside this disk.

come together at z_1 . The construction of P at ωz_1 and $\omega^2 z_1$ follows by symmetry and the construction at the other branch points is of a standard form, see e.g. [7].

The set $D(z_1, r) \setminus \Sigma_1$ consists of three parts, that we call regions I, II, III, as shown in Figure 12. Note that by (4.16) we have that the function

$$\widehat{\varphi}_1(z) = \begin{cases} \varphi_1(z) & \text{in region I,} \\ 2\pi i\beta - \varphi_1(z) & \text{in region II,} \\ -2\pi i\beta - \varphi_1(z) & \text{in region III,} \end{cases}$$

is analytic in $D(z_1, r) \setminus [0, z_1]$, By (4.12) it is real for real $z > z_1$ and for some constant $c > 0$,

$$\widehat{\varphi}_1(z) = c(z - z_1)^{3/2} + O(z - z_1)^{1/2} \quad \text{as } z \rightarrow z_1,$$

see (3.24). It then follows that

$$f_1(z) = \left[\frac{3}{2} \widehat{\varphi}_1(z) \right]^{2/3}, \quad z \in D(z_1, r) \quad (4.29)$$

is a conformal map from the disk $D(z_1, r)$ to a neighborhood of the origin, in such a way that $\Sigma_1 \cap D(z_1, r)$ is mapped by f_1 into the three half rays with angles $\pm\pi/3$ and π . We have the freedom to open the lenses around Σ_1 in such a way that the lips of the lenses in $D(z_1, r)$ are mapped by f_1 into half rays as well. In order to reduce the number of jump contours we open the lenses such that the lips of two consecutive lenses coincide in the disk $D(z_1, r)$ and are mapped by f_1 into the half rays with the angles 0 and $\pm 2\pi/3$. Note

that

$$\varphi_1(z) = \begin{cases} \frac{2}{3}f_1(z)^{3/2} & \text{in region I,} \\ -\frac{2}{3}f_1(z)^{3/2} + 2\pi i\beta & \text{in region II,} \\ -\frac{2}{3}f_1(z)^{3/2} - 2\pi i\beta & \text{in region III,} \end{cases}$$

and we find from (4.22) and (4.24), after some calculations, that the jump matrices for P can be written as,

$$J_P = J_S = \left\{ \begin{array}{ll} \begin{pmatrix} e^{n\varphi_{1,-}} & 0 & 0 \\ 0 & e^{-n\varphi_{1,-}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-n\varphi_{1,+}} & 0 & 0 \\ 0 & e^{n\varphi_{1,+}} & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } \arg f_1(z) = \pi \\ \begin{pmatrix} e^{n\varphi_{1,-}} & 0 & 0 \\ 0 & e^{-n\varphi_{1,-}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-n\varphi_{1,+}} & 0 & 0 \\ 0 & e^{n\varphi_{1,+}} & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } \arg f_1(z) = \pm\pi/3 \\ \begin{pmatrix} e^{n\varphi_1} & 0 & 0 \\ 0 & e^{-n\varphi_1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-n\varphi_1} & 0 & 0 \\ 0 & e^{n\varphi_1} & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } \arg f_1(z) = \pm 2\pi/3 \\ \begin{pmatrix} e^{n\varphi_1} & 0 & 0 \\ 0 & e^{-n\varphi_1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-n\varphi_1} & 0 & 0 \\ 0 & e^{n\varphi_1} & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } \arg f_1(z) = 0. \end{array} \right. \quad (4.30)$$

Here the contours that are mapped to angles $\pm 2\pi/3$ and π are oriented towards z_1 and the others are oriented away from z_1 .

We then look for a matrix valued function Ψ defined and analytic in an auxiliary ζ plane except with cuts at the half rays $\arg \zeta = k\pi/3$ for $k = -2, \dots, 3$, such that

$\Psi_+ = \Psi_- J_\Psi$ with

$$J_\Psi = \begin{cases} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } \arg \zeta = \pi, \\ \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } \arg \zeta = \pm\pi/3, \\ \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } \arg \zeta = 2\pi/3, \\ \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } \arg \zeta = 0. \end{cases} \quad (4.31)$$

Then for any analytic prefactor E_n we will have that

$$E_n(z) \Psi(n^{2/3} f_1(z)) \begin{pmatrix} e^{-n\varphi_1(z)} & 0 & 0 \\ 0 & e^{n\varphi_1(z)} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

satisfies the required jumps for the local parametrix.

The matrix Ψ is constructed with the Airy functions

$$y_0(\zeta) = \text{Ai}(\zeta), \quad y_1(\zeta) = \omega \text{Ai}(\omega\zeta), \quad y_2(\zeta) = \omega^2 \text{Ai}(\omega^2\zeta)$$

in the following way

$$\Psi(\zeta) = \begin{cases} \begin{pmatrix} -2y_2 & \frac{1}{2}y_0 & 0 \\ -2y'_2 & \frac{1}{2}y'_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{for } 0 < \arg \zeta < \pi/3, \\ \begin{pmatrix} -y_0 & -y_2 & 0 \\ -y'_0 & -y'_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{for } \pi/3 < \arg \zeta < 2\pi/3, \\ \begin{pmatrix} y_1 & -y_2 & 0 \\ y'_1 & -y'_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{for } 2\pi/3 < \arg \zeta < \pi, \\ \begin{pmatrix} 2y_1 & \frac{1}{2}y_0 & 0 \\ 2y'_1 & \frac{1}{2}y'_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{for } -\pi/3 < \arg \zeta < 0, \\ \begin{pmatrix} y_0 & -y_1 & 0 \\ y'_0 & -y'_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{for } -2\pi/3 < \arg \zeta < -\pi/3, \\ \begin{pmatrix} -y_2 & -y_1 & 0 \\ -y'_2 & -y'_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{for } -\pi < \arg \zeta < -2\pi/3. \end{cases}$$

From the known asymptotic behavior of the Airy functions we find the behavior

$$\begin{aligned} \Psi(n^{2/3}f_1(z)) \begin{pmatrix} e^{-n\varphi_1(z)} & 0 & 0 \\ 0 & e^{n\varphi_1(z)} & 0 \\ 0 & 0 & 1 \end{pmatrix} &= \frac{1}{2\sqrt{\pi}} \begin{pmatrix} n^{-1/6}f_1(z)^{-1/4} & 0 & 0 \\ 0 & n^{1/6}f_1(z)^{1/4} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &\times \begin{cases} \begin{pmatrix} 2i & \frac{1}{2} & 0 \\ 2i & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } -\pi/3 < \arg f_1(z) < \pi/3, \\ \begin{pmatrix} -1 & i & 0 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-2\pi i n \beta} & 0 & 0 \\ 0 & e^{2\pi i n \beta} & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } \pi/3 < \arg f_1(z) < \pi, \\ \begin{pmatrix} 1 & -i & 0 \\ -1 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{2\pi i n \beta} & 0 & 0 \\ 0 & e^{-2\pi i n \beta} & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } -\pi < \arg f_1(z) < -\pi/3. \end{cases} \\ &\times (I + O(n^{-1})) \quad \text{as } n \rightarrow \infty \quad (4.32) \end{aligned}$$

uniformly for z on the circle $|z - z_1| = r$.

The matching with M_n is then provided if we define

$$\begin{aligned} E_n(z) = \sqrt{\pi} M_n(z) \times &\begin{cases} \begin{pmatrix} -\frac{1}{2}i & -\frac{1}{2}i & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } -\pi/3 < \arg f_1(z) < \pi/3 \\ \begin{pmatrix} e^{2\pi i n \beta} & 0 & 0 \\ 0 & e^{-2\pi i n \beta} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ -i & -i & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } \pi/3 < \arg f_1(z) < \pi \\ \begin{pmatrix} e^{-2\pi i n \beta} & 0 & 0 \\ 0 & e^{2\pi i n \beta} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ i & i & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } -\pi < \arg f_1(z) < -\pi/3 \end{cases} \\ &\times \begin{pmatrix} n^{1/6}f_1(z)^{1/4} & 0 & 0 \\ 0 & n^{-1/6}f_1(z)^{-1/4} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4.33) \end{aligned}$$

It is straightforward to check that E_n is analytic in a full neighborhood of z_1 .

Then if we define

$$P(z) = E_n(z)\Psi(n^{2/3}f_1(z)) \begin{pmatrix} e^{-n\varphi_1(z)} & 0 & 0 \\ 0 & e^{n\varphi_1(z)} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we get $PM_n^{-1} = M_n(I + O(1/n))M_n^{-1} = I + M_nO(1/n)M_n^{-1}$ as $n \rightarrow \infty$. If $n \rightarrow \infty$ with $n \in \mathbb{N}_\varepsilon$ then by Proposition 4.7 (b), both M_n and M_n^{-1} are uniformly bounded on the circle $|z - z_1| = r$. This leads to the matching condition $PM_n^{-1} = I + O(1/n)$ as $n \in \mathbb{N}_\varepsilon \rightarrow \infty$, uniformly for $z \in \partial D(z_1, r)$ as required in the RH problem 4.8.

4.4 Proof of Theorem 2.4

In the final transformation we define

$$R(z) = \begin{cases} S(z)P(z)^{-1} & \text{in } D, \\ S(z)M(z)^{-1} & \text{elsewhere.} \end{cases} \quad (4.34)$$

Recall that D denotes the union of nine disks $D(\omega^j z_k)$, $j = 0, 1, 2$, $k = 1, 2, 3$.

Then R is defined and analytic outside $\Gamma_S \cup \partial D$ and it has an analytic extension across Σ_1 , Σ_2 and across the parts of Γ_S that are in D . Thus R is analytic in $\mathbb{C} \setminus \Gamma_R$ where Γ_R consists of ∂D , and the parts of $\Gamma_S \setminus (\Sigma_1 \cup \Sigma_2)$ that are outside the disks, and we have the following RH problem.

RH problem 4.9. • R is analytic in $\mathbb{C} \setminus \Gamma_R$.

- $R_+ = R_- J_R$ on Γ_R where

$$J_R(z) = \begin{cases} M(z)^{-1}P(z) & \text{for } z \text{ on the circles,} \\ M(z)^{-1}J_S(z)M(z) & \text{elsewhere on } \Gamma_R. \end{cases}$$

- $R(z) = I + O(z^{-1})$ as $z \rightarrow \infty$.

As a result of the matching condition in the RH problem 4.8 for P we have

$$J_R(z) = I + O(1/n) \quad \text{as } n \rightarrow \infty, n \in \mathbb{N}_\varepsilon.$$

On the remaining parts of Γ_R the jumps for R are exponentially close to the identity matrix:

$$J_R(z) = I + O(e^{-cn}) \quad \text{elsewhere on } \Gamma_R$$

for some $c > 0$. This follows from the formulas (4.22) and (4.24) for the jump matrices J_S which are of the form $I + O(e^{-cn})$ and we also use Proposition 4.7 (b), which says that $M(z)$ and $M(z)^{-1}$ are uniformly bounded on Γ_R . For $z \rightarrow \infty$, the estimate can be sharpened to

$$J_R(z) = I + O(e^{-cn|z|^3}) \quad \text{elsewhere on } \Gamma_R.$$

Then by standard estimates on RH problems [7], we have

$$R(z) = I + O\left(\frac{1}{n(1+|z|)}\right) \quad \text{as } n \rightarrow \infty, n \in \mathbb{N}_\varepsilon \quad (4.35)$$

uniformly for $z \in \mathbb{C} \setminus \Gamma_R$. The estimate (4.35) is the final result of the steepest descent analysis of the RH problem.

Proof of Theorem 2.4. We can now prove Theorem 2.4 by following the steps $Y \mapsto X \mapsto V \mapsto U \mapsto T \mapsto S \mapsto R$ to see the effect on the polynomial $P_{n,n}$. This is similar to the proof of Lemma 6.1 in [5]. As in that proof we find

$$P_{n,n}(z) = S_{11}(z)e^{ng_1(z)}, \quad z \in \mathbb{C} \setminus L_1$$

where L_1 denotes the lens around Σ_1 . Also $S = RM$ with R satisfying (4.35) gives us

$$S(z) = M_{n,11}(z) + O(1/n), \quad z \in \mathbb{C} \setminus L_1, n \in \mathbb{N}_\varepsilon.$$

where we also used the fact that $M_{n,11}(z)$ remains bounded, see Proposition 4.7. This proves (2.34) with a $O(1/n)$ term that is uniform for $z \in \mathbb{C} \setminus L_1$. Since we have the freedom to open the lens as small as we wish, we find (2.34) with $M_{n,11}$ given by (2.36) according to part (c) of Proposition 4.7.

The $O(1/n)$ is also uniform for t in compact subsets of t_*, t_{**} , for values of $n \in \mathbb{N}_\varepsilon$, where \mathbb{N}_ε , see (2.33), is varying with t .

This completes the proof of Theorem 2.4, pending the proof of Proposition 4.7 that will follow in the next section. \square

5 Proof of Proposition 4.7

5.1 Riemann surface \mathcal{S}

To solve the model RH problem we use the Riemann surface \mathcal{R} as before and with the sheet structure shown in Figure 2. It has genus three.

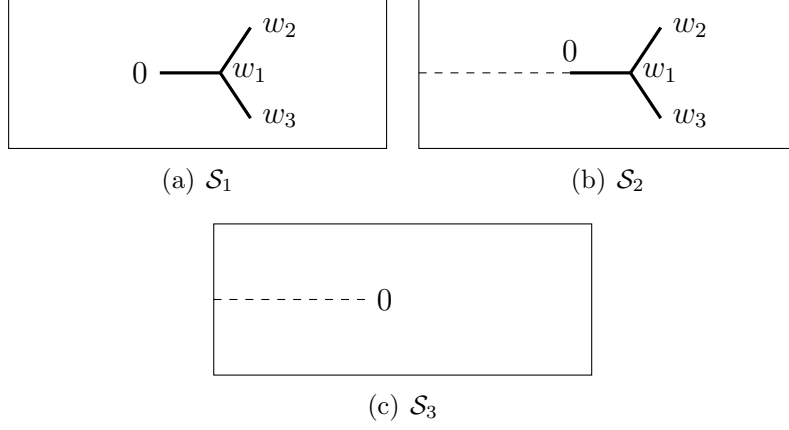


Figure 13: The three sheets \mathcal{S}_1 , \mathcal{S}_2 and \mathcal{S}_3 of the Riemann surface \mathcal{S}

The Riemann surface has three fold symmetry

$$\rho : \mathcal{R} \rightarrow \mathcal{R} : (z, \xi) \mapsto (\omega z, \omega^2 \xi) \quad (5.1)$$

which induces an action of \mathbb{Z}_3 on \mathcal{R} . It will be useful to consider the orbit space which is a Riemann surface that we call \mathcal{S} , and the quotient map

$$\psi : \mathcal{R} \rightarrow \mathcal{S} : (z, \xi) \mapsto (z^3, z\xi). \quad (5.2)$$

Since \mathcal{R} is defined by the equation (2.1), we find that \mathcal{S} has the equation (where $w = z^3$ and $\eta = z\xi$),

$$\mathcal{S} : \quad \eta^3 - w\eta^2 - (1+t)w\eta + w^2 + Aw = 0. \quad (5.3)$$

Then \mathcal{S} is a genus one Riemann surface, whose sheet structure is shown in Figure 13 with $w_j = z_j^3$ for $j = 1, 2, 3$.

The branch points are connected by cuts that are the images of the cuts $\gamma_{1,2}$ and $\gamma_{1,3}$ under the mapping $w = z^3$. We denote these by

$$[w_1, w_2] := \gamma_{1,2}^3, \quad [w_1, w_3] := \gamma_{1,3}^3 \quad (5.4)$$

but we emphasize that these are not exact straight line segments.

There are three solutions to (5.3) with respective asymptotic behaviors

$$\begin{aligned} \eta_1(w) &= w + t + O(w^{-1}) \\ \eta_2(w) &= w^{1/2} - \frac{t}{2} + O(w^{-1/2}) \\ \eta_3(w) &= -w^{1/2} - \frac{t}{2} + O(w^{-1/2}) \end{aligned} \quad (5.5)$$

as $w \rightarrow \infty$. The solution η_j is defined and analytic on sheet \mathcal{S}_j for $j = 1, 2, 3$. Note that $\eta_j(z^3) = z\xi_j(z)$.

The Riemann surface \mathcal{S} has an anti-holomorphic involution $\sigma : \mathcal{S} \rightarrow \mathcal{S} : (w, \eta) \mapsto (\bar{w}, \bar{\eta})$. The real part of \mathcal{S} is

$$\mathcal{S}_{real} = \{Q \in \mathcal{S} \mid \sigma(Q) = Q\}. \quad (5.6)$$

which is a closed loop that contains the two points at infinity. We provide it with an orientation from left to right on the intervals $[w_1, \infty)$ and $(-\infty, 0]$ on the first sheet, and the interval $[0, \infty)$ on the third sheet, and from right to left on the interval $(\infty, w_1]$ on the second sheet. This cycle is called c_S . Also

$$c_S = \psi(c_R)$$

where c_R is the cycle on \mathcal{R} that was introduced in section 2.2.

There is a cycle b_S that starts and ends at w_1 and goes around $[w_1, w_2]$ on the first sheet in counterclockwise fashion. The cycle \bar{b}_S goes around $[w_1, w_3]$ on the first sheet with clockwise orientation. Then (\bar{b}_S, b_S) is a canonical homology basis for \mathcal{S} and

$$c_S = b_S + \bar{b}_S \quad (5.7)$$

with equality in the sense of homotopic cycles. Also

$$a_S = b_S - \bar{b}_S \quad (5.8)$$

is a cycle that goes around both $[w_1, w_2]$ and $[w_1, w_3]$ on the first sheet with counterclockwise orientation.

Since the genus of \mathcal{S} is one, there is a unique holomorphic differential ω_S such that

$$\oint_{c_S} \omega_S = 1. \quad (5.9)$$

Using the equation (5.3) for \mathcal{S} we can find

$$\omega_S = \frac{C dw}{3\eta^2 - 2w\eta - (1+t)w} \quad (5.10)$$

for some constant $C > 0$. The denominator $3\eta^2 - 2w\eta - (1+t)w$ has simple zeros at the branch points w_1, w_2, w_3 and a double zero at 0. It is also real on \mathcal{S}_{real} and in fact positive on the real parts of the first and third sheets, and negative on the real part of the second sheet, as can be verified from (5.5) and the fact that there are no other zeros on \mathcal{S}_{real}

besides w_1 and 0. A local analysis shows that (5.10) is indeed holomorphic at the branch points, as well as at the points at infinity.

Note that the holomorphic differential ω_R from (2.27) is the pullback of ω_S onto \mathcal{R} :

$$\psi^*(\omega_S) = \omega_R.$$

Because of symmetry, the a_S period of ω_S is purely imaginary, with positive imaginary part (due to the chosen orientation). We introduce

$$\tau := \oint_{a_S} \omega_S \in i\mathbb{R}^+ \quad (5.11)$$

which is the same number as given by (2.28). Then by (5.7)–(5.8) and (5.9)–(5.11)

$$\oint_{b_S} \omega_S = \frac{1}{2} + \frac{1}{2}\tau, \quad \oint_{\bar{b}_S} \omega_S = \frac{1}{2} - \frac{1}{2}\tau. \quad (5.12)$$

The lattice L of periods is thus generated by $\frac{1}{2} - \frac{1}{2}\tau$ and $\frac{1}{2} + \frac{1}{2}\tau$.

$$L = \{m(\frac{1}{2} - \frac{1}{2}\tau) + n(\frac{1}{2} + \frac{1}{2}\tau) \mid m, n \in \mathbb{Z}\}. \quad (5.13)$$

The Abel map with base point ∞_1 (the point at infinity on \mathcal{S}_1) is

$$u : \mathcal{S} \rightarrow \mathbb{C}/L : Q \in \mathcal{S} \mapsto \int_{\infty_1}^Q \omega_S \quad \text{modulo periods} \quad (5.14)$$

and it identifies \mathcal{S} with the complex torus \mathbb{C}/L . A fundamental domain for \mathbb{C}/L is the parallelogram with vertices $0, \frac{1}{2} - \frac{1}{2}\tau, 1, \frac{1}{2} + \frac{1}{2}\tau$. The Abel map is real-valued on c_S .

To make the Abel map single valued we take away the cuts $[w_1, w_2]$ and $[w_1, w_3]$ (that is, the a_S cycle), and the restrictions of the Abel map are then denoted by u_1, u_2, u_3 . Thus

$$u_j(w) = u(w^{(j)}) = \int_{\infty_1}^{w^{(j)}} \omega_S, \quad j = 1, 2, 3 \quad (5.15)$$

where $w^{(j)}$ is the point on the j th sheet that projects onto $w \in \mathbb{C}$ and the path of integration does not intersect $[w_1, w_2]$ and $[w_1, w_3]$ on the sheets \mathcal{S}_1 and \mathcal{S}_2 . Then u_1 is defined and analytic on $\overline{\mathbb{C}} \setminus ([0, w_1] \cup [w_1, w_2] \cup [w_1, w_3])$ with $u_1(\infty) = 0$, u_2 is analytic on $\mathbb{C} \setminus ((-\infty, w_1] \cup [w_1, w_2] \cup [w_1, w_3])$, u_3 is analytic on $\mathbb{C} \setminus (-\infty, 0]$ with the following properties

$$u_{1,\pm} = u_{2,\mp} \quad \text{on } [0, w_1] \quad (5.16)$$

$$u_{1,\pm}(w) - u_{2,\mp}(w) = - \oint_{\bar{b}_S} \omega_S = -\frac{1}{2}(1 - \tau) \quad \text{on } [w_1, w_2] \quad (5.17)$$

$$u_{1,\pm}(w) - u_{2,\mp}(w) = - \oint_{b_S} \omega_S = -\frac{1}{2}(1 + \tau) \quad \text{on } [w_1, w_3], \quad (5.18)$$

see (5.12), and

$$u_{2,\pm} = u_{3,\mp} \quad \text{on } (-\infty, 0]. \quad (5.19)$$

We also note that $u_1 + u_2 + u_3$ is constant on \mathbb{C} , since this sum is bounded and analytic outside of $(-\infty, w_1]$, $[w_1, w_2]$ and $[w_1, w_3]$, and has no jumps on any of these arcs as follows from (5.16)–(5.19).

5.2 First step

The goal of the first step is to remove the prefactors 2 and 1/2 in the offdiagonal entries of the jump matrices on the arcs $\bigcup_j \omega^j \gamma_{1,2}$ and $\bigcup_j \omega^j \gamma_{1,3}$, see (4.26). Here we use the components u_1, u_2, u_3 of the Abel map of \mathcal{S} evaluated in z^3 . We seek M in the form

$$M(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^{2u_2(\infty)} & 0 \\ 0 & 0 & 2^{2u_3(\infty)} \end{pmatrix} N(z) \begin{pmatrix} 2^{-2u_1(z^3)} & 0 & 0 \\ 0 & 2^{-2u_2(z^3)} & 0 \\ 0 & 0 & 2^{-2u_3(z^3)} \end{pmatrix}. \quad (5.20)$$

Note that $u_2(\infty) = u_3(\infty)$.

In order that M satisfies the RH problem for M we need that N satisfies the following RH problem. The jumps in (5.21) are obtained from (4.26), (5.16)–(5.19), and the definition (5.20) of N in terms of M .

RH problem 5.1. • N is analytic in $\mathbb{C} \setminus (\Sigma_1 \cup \Sigma_2)$.

- $N_+ = N_- J_N$ on $\Sigma_1 \cup \Sigma_2$ with

$$J_N = \begin{cases} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{on } \Sigma_1^o \\ \begin{pmatrix} 0 & 2^{-\tau} e^{2\pi i n \beta} & 0 \\ -2^\tau e^{-2\pi i n \beta} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{on } \bigcup_{j=0}^2 \omega^j \gamma_{1,2} \\ \begin{pmatrix} 0 & 2^\tau e^{-2\pi i n \beta} & 0 \\ -2^{-\tau} e^{2\pi i n \beta} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{on } \bigcup_{j=0}^2 \omega^j \gamma_{1,3} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} & \text{on } \Sigma_2. \end{cases} \quad (5.21)$$

- $N(z) = (I + O(1/z))A(z)$ as $z \rightarrow \infty$ where $A(z)$ is given by (4.7).
- For $j, k = 1, 2, 3$ we have

$$N(z) = O\left((z - \omega^j z_k)^{-1/4}\right) \quad \text{as } z \rightarrow \omega^j z_k. \quad (5.22)$$

To see that the asymptotic condition in the RH problem for N is indeed the same as the one for M requires some calculations and uses the facts that $u_1(z^3) = O(z^{-3})$, $u_2(z^3) = u_2(\infty) + O(z^{-3/2})$, and $u_3(z^3) = u_3(\infty) + O(z^{-3/2})$ as $z \rightarrow \infty$ with $u_2(\infty) = u_3(\infty)$.

The effect of the first step is that the RH problem for N now depends on the real parameter (recall that τ is purely imaginary),

$$\nu = \nu_n = n\beta - \frac{\tau}{2\pi i} \log 2 \quad (5.23)$$

and so

$$J_N = \begin{cases} \begin{pmatrix} 0 & e^{2\pi i \nu} & 0 \\ -e^{-2\pi i \nu} & 0 & 0 \end{pmatrix} & \text{on } \bigcup_{j=0}^2 \omega^j \gamma_{1,2} \\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & e^{-2\pi i \nu} & 0 \\ -e^{2\pi i \nu} & 0 & 0 \end{pmatrix} & \text{on } \bigcup_{j=0}^2 \omega^j \gamma_{1,3}. \end{cases} \quad (5.24)$$

We consider $\nu \in \mathbb{R}/\mathbb{Z}$.

5.3 Second step

Our next task is to construct functions v_1, v_2, v_3 that are defined and holomorphic on the respective sheets $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ of \mathcal{S} such that

$$(v_1 \ v_2 \ v_3)_+ = (v_1 \ v_2 \ v_3)_- \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } [0, w_1] \quad (5.25)$$

$$(v_1 \ v_2 \ v_3)_+ = (v_1 \ v_2 \ v_3)_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \text{on } (-\infty, 0] \quad (5.26)$$

$$(v_1 \ v_2 \ v_3)_+ = (v_1 \ v_2 \ v_3)_- \begin{pmatrix} 0 & e^{2\pi i \nu} & 0 \\ -e^{-2\pi i \nu} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } [w_1, w_2], \quad (5.27)$$

$$(v_1 \ v_2 \ v_3)_+ = (v_1 \ v_2 \ v_3)_- \begin{pmatrix} 0 & e^{-2\pi i \nu} & 0 \\ -e^{2\pi i \nu} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } [w_1, w_3] \quad (5.28)$$

$$(v_1, v_2, v_3) = \begin{cases} O((w - w_k)^{-1/4}) & \text{as } w \rightarrow w_k \text{ for } k = 1, 2, 3, \\ O(1) & \text{as } w \rightarrow 0, \end{cases} \quad (5.29)$$

$$v_1 = O(1), \quad v_2 = O(w^{-1/4}), \quad v_3 = O(w^{-1/4}) \text{ as } w \rightarrow \infty. \quad (5.30)$$

If we can find such functions v_j then $(v_1(z^3), v_2(z^3), v_3(z^3))$ is a vector that satisfies $v_+ = v_- J_N$.

The problem for v clearly depends on ν . In the second step we show that it is possible to solve it for the particular value $\nu = 1/2$. In this case (and also in the case $\nu = 0$) all non-zero entries in the jump matrices in (5.25)–(5.28) are ± 1 . This then implies that any solution (v_1, v_2, v_3) of the above vector valued RH problem, yields a meromorphic function on \mathcal{S} defined by

$$Q = (w, \eta) \in \mathcal{S} \mapsto v_j^2(w), \quad \text{if } Q \in \mathcal{S}_j \text{ for } j = 1, 2, 3.$$

Because of (5.29) this function can have simple poles at w_1, w_2, w_3 , but not at $w = 0$, and because of (5.30) it has a simple zero at ∞_2 . Having three poles, the function must have two more zeros, and the only possibility is to have a double zero somewhere in \mathcal{S} .

We can now solve the problem for v by reversing the arguments. We start by noting that the function

$$F : \mathcal{S} \rightarrow \overline{\mathbb{C}} : \quad (w, \eta) \in \mathcal{S} \mapsto \frac{\eta^2}{3\eta^2 - 2w\eta - (1+t)w} \quad (5.31)$$

is meromorphic with simple poles at $w = w_1, w_2, w_3$, a simple zero at ∞_2 (see also the asymptotics (5.5)) and a double zero at

$$Q_{1/2} := (-A, 0) \in \mathcal{S}_1.$$

Indeed, by (5.3) we have that $\eta = 0$ implies $w = 0$ or $w = -A$. The numerator η^2 in (5.31) thus gives the double zero at $Q_{1/2}$ and an inspection of the behavior of η_1 reveals that $Q_{1/2} \in \mathcal{S}_1$. The origin $(w, \eta) = (0, 0)$ is not a zero of F since the denominator of (5.31) also vanishes quadratically at the origin and $F(0, 0) = \frac{1}{3} > 0$. Also $F(\infty_1) = 1$.

Let F_j denote the restriction of F to the j th sheet. Since each sheet is simply connected and the zero at $Q_{1/2} \in \mathcal{S}_1$ is a double zero (the other zeros and poles are on the cuts), we can take an analytic square root on each sheet. We do it in such a way that $v_j^2 = F_j$ with $v_1(\infty) = 1$, $v_1(0) = -\frac{1}{\sqrt{3}}$, $v_{2,+}(0) = -v_{2,-}(0) = \frac{1}{\sqrt{3}}$ and $v_3(0) = \frac{1}{\sqrt{3}}$. The construction is such that (5.25) and (5.26) are satisfied.

Then a careful analysis about how the branches of the square root behave under the analytic continuation (we were assisted by Maple) shows that the jump matrix on $[w_1, w_2]$

and $[w_1, w_3]$ is $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which is the jump matrix in (5.27) and (5.28) with $\nu = 1/2$.

Thus we can solve the vector problem for the parameter $\nu = 1/2$. The solution is denoted by $(v_1^{(1/2)}, v_2^{(1/2)}, v_3^{(1/2)})$. Note that $v_1^{(1/2)}$ has a simple zero at the point $-A$ on the negative real line. We also have $v_1^{(1/2)}(\infty) = 1$ and $v_2^{(1/2)}(w) = O(w^{-1/4})$, $v_3^{(1/2)}(w) = O(w^{-1/4})$, as $w \rightarrow \infty$.

The construction gives in particular by (5.31)

$$v_1^{(1/2)}(w) = F^{1/2}(w^{(1)}) = \frac{\eta_1(w)}{(3\eta_1(w)^2 - 2w\eta_1(w) - (1+t)w)^{1/2}}. \quad (5.32)$$

Remark 5.2. There is a constant $c \in (0, 1)$ such that

$$\tilde{F} : \mathcal{S} \rightarrow \overline{\mathbb{C}} : (w, \eta) \mapsto \frac{\eta^2 - cw}{3\eta^2 - 2w\eta - (1+t)w}$$

has a double zero at a point $Q_0 \in \mathcal{S}_2 \cap \mathcal{S}_{real}$. It further has simple poles at w_1, w_2, w_3 and a simple zero at ∞_2 . Then a similar construction, yields a vector $(v_1^{(0)}, v_2^{(0)}, v_3^{(0)})$ that satisfies the jump conditions in the vector problem with $\nu = 0$. The difference in jumps comes from the fact that $(\eta^2 - cw)^{1/2}$ is not a globally defined analytic function on the Riemann surface (despite having only double zeros and poles). The change in argument of $\eta^2 - cw$ along the b_S and \bar{b}_S cycle is an odd multiple of 2π , and then the change in argument of $(\eta^2 - cw)^{1/2}$ is by an odd multiple of π , which leads to a change in sign in the jump matrices.

5.4 Third step

We start from the functions $v_1^{(1/2)}$, $v_2^{(1/2)}$ and $v_3^{(1/2)}$ that solve the vector problem with $\nu = 1/2$. We are going to use Jacobi theta functions to modify the functions so that they solve the vector problem with an arbitrary $\nu \in [0, 1]$, see also [6, 8] for similar uses of theta functions in RH problems. Let $\theta(s)$ be the theta function as in (2.29) which has zeros at the values $s_0 \pmod{L}$ with s_0 as in (2.31) and no other zeros. Let

$$\delta = u(Q_{1/2}) - s_0 = \int_{-\infty_1}^{-A} \omega_S - \frac{-1 + \tau}{4} \quad (5.33)$$

Then $Q \mapsto \theta(u(Q) - \delta)$ has a simple zero at $Q_{1/2}$ (and no other zeros) and it follows that the functions

$$v_j^{(\nu)}(w) = \frac{\theta(u_j(w) - \delta + \nu - 1/2)}{\theta(u_j(w) - \delta)} v_j^{(1/2)}(w), \quad w \in \mathcal{S}_j, \quad j = 1, 2, 3 \quad (5.34)$$

are well-defined and analytic. The zero of $\theta(u_1(w) - \delta)$ at $w = -A$ is cancelled by the zero of $v_1^{(1/2)}(w)$.

Lemma 5.3. *The vector $(v_1^{(\nu)}, v_2^{(\nu)}, v_3^{(\nu)})$ satisfies the conditions for the vector problem (5.25)–(5.30).*

Proof. The ratio of theta functions

$$\Theta(s) := \frac{\theta(s - \delta + \nu - 1/2)}{\theta(s - \delta)} \quad (5.35)$$

has periodicity properties

$$\Theta(s + 1) = \Theta(s), \quad \Theta(s \pm \frac{1+\tau}{2}) = -e^{\mp 2\pi i \nu} \Theta(s), \quad \Theta(s \pm \frac{1-\tau}{2}) = -e^{\pm 2\pi i \nu} \Theta(s),$$

which easily follows from (2.30) and (5.35).

Let $w \in [w_1, w_2]$. Then $v_{1,+}^{(1/2)}(w) = v_{2,-}^{(1/2)}(w)$ and $u_{1,+}(w) = u_{2,-}(w) + \frac{-1+\tau}{2}$ by (5.17) so that by the periodicity property

$$\Theta(u_{1,+}(w)) = \Theta(u_{2,-}(w) - \frac{1-\tau}{2}) = -e^{-2\pi i \nu} \Theta(u_{2,-}(w)).$$

Thus

$$\begin{aligned} v_{1,+}^{(\nu)}(w) &= \Theta(u_{1,+}(w)) v_{1,+}^{(1/2)}(w) \\ &= -e^{-2\pi i \nu} \Theta(u_{2,-}(w)) v_{2,-}^{(1/2)}(w) \\ &= -e^{-2\pi i \nu} v_{2,-}^{(\nu)}(w). \end{aligned}$$

Similarly, $v_{2,+}^{(\nu)}(w) = e^{2\pi i\nu} v_{1,-}^{(\nu)}(w)$, which gives the jump (5.27) on $[w_1, w_2]$. A similar calculation shows that $v_1^{(\nu)}, v_2^{(\nu)}, v_3^{(\nu)}$ satisfies the jump (5.28) on $[w_1, w_3]$. The jumps on $[0, w_1]$ and $(-\infty, 0]$, as well as the asymptotic conditions (5.29) and (5.30) are straightforward to verify. \square

By Lemma 5.3 we can solve the vector problem for any real ν . The solution is not unique, since we can multiply by a common constant. If $v_1^{(\nu)}(\infty) \neq 0$, then we can normalize the solution and we find that

$$\begin{aligned} N_{1,j}(z) &= \frac{v_j^{(\nu)}(z^3)}{v_1^{(\nu)}(\infty)} \\ &= \frac{\theta(-\delta)}{\theta(-\delta + \nu - 1/2)} \frac{\theta(u_j(z^3) - \delta + \nu - 1/2)}{\theta(u_j(z^3) - \delta)} \frac{v_j^{(1/2)}(z^3)}{v_1^{(1/2)}(\infty)} \end{aligned} \quad (5.36)$$

gives a vector $(N_{1,j}, N_{2,j}, N_{3,j})$ that satisfies the conditions for the first row in the RH problem (5.1).

This fails if $v_1^{(\nu)}(\infty) = 0$, which by (5.34) and the fact that $u_1(\infty) = 0$ comes down to $\theta(-\delta + \nu - 1/2) = 0$. Thus $-\delta + \nu - 1/2 \equiv s_0 \pmod{L}$. From (5.33) it then follow that there is unique $\nu = \nu^*$ for which this holds, namely

$$\nu^* = \frac{1}{2} + \int_{-\infty_1}^{-A} \omega_S \pmod{\mathbb{Z}}.$$

Using $\psi^*(\omega_S) = \omega_R$, we can also write

$$\nu^* = \frac{1}{2} + \int_{-\infty_1}^{-A^{1/3}} \omega_R \pmod{\mathbb{Z}}, \quad (5.37)$$

where $-A^{1/3}$ denotes the point $(-A^{1/3}, 0)$ that is on the first sheet \mathcal{R}_1 of the Riemann surface.

In view of (5.23) and (5.37) we conclude that if

$$n\beta \not\equiv \beta^* \pmod{\mathbb{Z}}, \quad (5.38)$$

where β^* is given by (4.27), then (5.36) solves the first row in the RH problem for N . It also follows that if, $n \in \mathbb{N}_\varepsilon$, see (2.33), then the distance from $n\beta - \nu^* - \frac{\tau}{2\pi i} \log 2$ to the set of integers is at least ε , and then the entries (5.36) are uniformly bounded for z in compact subsets of $\overline{\mathbb{C}}$ away from the branch points with a bound that only depends on ε .

5.5 Fourth step

Before we can continue with filling in the other rows of N , we need a lemma. Recall that ρ is the symmetry (5.1) of the Riemann surface \mathcal{R} . The real part of \mathcal{R} is

$$\mathcal{R}_{real} = \{(\xi, z) \in \mathbb{R} \times \mathbb{R} \mid P(\xi, z) = 0\}$$

where P is the algebraic equation (2.1) for \mathcal{R} .

Lemma 5.4. *Let $P \in \mathcal{R}_{real}$. Then the divisor $D = P + \rho(P) + \rho^2(P)$ is non-special.*

Proof. The space $L(D)$ contains all meromorphic functions on \mathcal{R} with poles only at P , ωP , $\omega^2 P$. These are all simple poles if $P \notin \{\infty_1, \infty_2\}$, and at most poles of order three otherwise. We have to show that $\dim L(D) = 1$, that is, the only functions in $L(D)$ are the constant functions.

The three points $P, \rho(P), \rho^2(P)$ are all mapped by (5.2) to the same point $Q \in \mathcal{S}$. Note that $\dim L(Q) = 1$, since there are no special points on a genus one Riemann surface. A meromorphic function \tilde{f} on S gives rise to a meromorphic function $f = \tilde{f} \circ \psi$ on \mathcal{R} which is invariant under the \mathbb{Z}_3 action, and any \mathbb{Z}_3 invariant meromorphic function can be obtained that way. It follows that $L(D)$ does not contain any \mathbb{Z}_3 invariant meromorphic functions, except for constants.

Let $f \in L(D)$. Then $f + f \circ \rho + f \circ \rho^2$ is \mathbb{Z}_3 invariant, and therefore a constant, say

$$f + f \circ \rho + f \circ \rho^2 = 3c \tag{5.39}$$

for some constant $c \in \mathbb{C}$. We now distinguish three cases $P \notin \{\infty_1, \infty_2\}$, $P = \infty_1$ and $P = \infty_2$.

Case $P \notin \{\infty_1, \infty_2\}$ In this case we have Laurent expansions of f about ∞_1 and ∞_2 of the form

$$f(z) = \begin{cases} c + c_1 z^{-1} + \cdots & \text{as } z \rightarrow \infty_1, \\ c + d_1 z^{-1} + \cdots & \text{as } z \rightarrow \infty_2 \end{cases}$$

where we recall that z^{-1} is the local coordinate around ∞_1 and $z^{-1/2}$ around ∞_2 . We do not have fractional exponents in the expansion about ∞_2 since that would be incompatible with (5.39). Then $z \mapsto z(f(z) - c)$ is holomorphic at both ∞_1 and ∞_2 and since we do not introduce any other poles,

$$f_1 = \pi_1(f - c) \in L(D), \quad \pi_1 : \mathcal{R} \rightarrow \widehat{\mathbb{C}} : (z, \xi) \mapsto z.$$

We iterate this argument, and inductively find a sequence (f_n) of functions and a sequence (c_n) of constants such that

$$f_{n+1} = \pi_1(f_n - c_n) \in L(D).$$

Since $L(D)$ is finite dimensional, there is a linear relations between f, f_1, \dots, f_n for some n . Then f is rational in z , which means that together with pole at P , it also poles at other points on the Riemann surface with the same z -coordinate. This forces f to be a constant.

Case $P = \infty_1$ In this case there is a possible pole at ∞_1 of order ≤ 3 and the Laurent expansions have the form

$$f(z) = \begin{cases} c_{-3}z^3 + c_{-2}z^2 + c_{-1}z + c + O(z-1) + \dots & \text{as } z \rightarrow \infty_1, \\ c + O(z^{-1}) + \dots & \text{as } z \rightarrow \infty_2. \end{cases}$$

The identity (5.39) implies $c_{-3} = 0$, and as in the first case we find

$$f_1 = \pi_1(f - c) \in L(D).$$

Then we can argue as above and conclude that f is a constant.

Case $P = \infty_2$ In this case there are expansions

$$f(z) = \begin{cases} c + c_1z^{-1} + O(z^{-2}) & \text{as } z \rightarrow \infty_1, \\ c_{-1}z + c + O(z^{-1}) & \text{as } z \rightarrow \infty_2, \end{cases}$$

since again there can be no terms with $z^{3/2}$, $z^{1/2}$ and $z^{-1/2}$ because of (5.39). If $c_{-1} \neq 0$, then $\pi_1(f - c)$ has a pole of order 4 at ∞_2 , and so it does not belong to $L(D)$, as in the other cases. However we now use that $f \circ \rho$ also belongs to $L(D)$ and has expansions in local coordinates

$$(f \circ \rho)(z) = \begin{cases} c + c_1\omega^2z^{-1} + O(z^{-2}) & \text{as } z \rightarrow \infty_1, \\ c_{-1}\omega z + c + O(z^{-1}) & \text{as } z \rightarrow \infty_2. \end{cases}$$

where we assume $c_{-1} \neq 0$ (otherwise f has no poles at all, and clearly is a constant). Then

$$g = \omega^2 f - f \circ \rho - \omega^2 c + c$$

is in $L(D)$ with

$$g(z) = \begin{cases} O(z^{-2}) & \text{as } z \rightarrow \infty_1, \\ c_{-1}(\omega^2 - \omega)z + O(z^{-1}) & \text{as } z \rightarrow \infty_2. \end{cases}$$

Thus g has a double pole at ∞_2 and a double zero at ∞_1 . There are no other zeros or poles, and so $1/g$ has a double pole at ∞_1 , which means that

$$1/g \in L(3\infty_1).$$

From the second case we already know that $L(3\infty_1)$ consists of constant functions only. Thus g is a constant, which is a contradiction with $c_{-1} \neq 0$. \square

We can now complete solution of N in the same way as in [5, section 6.6].

We assume that $\nu \neq \nu^*$. Then (5.36) gives us the first row of N . The ratio (5.35) of shifted theta-functions has a zero at $s = s_0 + \delta - \nu + 1/2 = u(Q_{1/2}) - \nu + 1/2$. There is a value Q_ν on the real part of \mathcal{S} with

$$u(Q_\nu) = u(Q_{1/2}) - \nu + 1/2 \quad \text{mod } \mathbb{Z}.$$

If $Q_\nu = (w, \eta_j(w))$ is on the sheet \mathcal{S}_j , then it follows from (5.34) that $v_j^\nu(w) = 0$, and hence $N_{1,j}(z) = 0$ whenever $z^3 = w$. Also $Q_\nu \neq \infty_1$, since $\nu \neq \nu^*$.

Then $\psi^{-1}(Q_\nu) = \{P_\nu, \rho(P_\nu), \rho^2(P_\nu)\}$ for some $P_\nu \in \mathcal{R}_{real} \setminus \{\infty_1\}$. The divisor

$$D = P_\nu + \rho(P_\nu) + \rho^2(P_\nu)$$

is non-special by Lemma 5.4. Thus $\dim L(D) = 1$. It then follows that $L(D + 2\infty_2)$ is three dimensional (by the Riemann-Roch theorem). It has a basis $\{1, f, g\}$. Let f_1, f_2, f_3 , and g_1, g_2, g_3 denote the restrictions of f and g to the respective sheets of \mathcal{S} , and put

$$B := \begin{pmatrix} N_{1,1} & N_{1,2} & N_{1,3} \\ f_1 N_{1,1} & f_2 N_{1,2} & f_3 N_{1,3} \\ g_1 N_{1,1} & g_2 N_{1,2} & g_3 N_{1,3} \end{pmatrix}. \quad (5.40)$$

If $P_\nu = (z, \xi_j(z)) \in \mathcal{R}_j$ then f_j and g_j have a possible pole at z , ωz , and $\omega^2 z$. However, the poles are compensated by the zero of $N_{1,j}$ and it follows that B is analytic in $\mathbb{C} \setminus (\Sigma_1 \cup \Sigma_2)$.

It is then easy to verify that B satisfies the jumps $B_+ = B_- J_N$ as in the RH problem 5.1 for N . Since $A_+ = A_- J_N$ on Σ_2 , we find that BA^{-1} is analytic across Σ_2 and therefore it is single valued at infinity. It can be verified that $B(z) = O(z^{1/4})$ and $A(z) = O(z^{1/4})$ as $z \rightarrow \infty$ which means that the Laurent expansion of BA^{-1} has the form

$$(BA^{-1})(z) = C + O(z^{-1}) \quad \text{as } z \rightarrow \infty \quad (5.41)$$

with a constant matrix C . From (4.7) we see that $\det A \equiv 1$. From $B_+ = B_- J_N$ where $\det J_N \equiv 1$ it also follows that $\det B \equiv b$ for some constant b . If the constant were zero, then we see from (5.40) that there is a nontrivial linearly combination $h = c_0 + c_1 f + c_2 g$ such that $h_j N_{1,j} \equiv 0$ for each j . The functions $N_{1,j}$ are analytic and they do not vanish identically, which implies that $h = 0$ and this is impossible since $\{1, f, g\}$ are linearly independent.

Then by (5.41) we have

$$\det C = \lim_{z \rightarrow \infty} \det(BA^{-1}(z)) = b \neq 0$$

and so C is an invertible matrix.

Then $N = C^{-1}B$ satisfies the jump conditions (5.21) in the RH problem for N . The asymptotic condition is satisfied because by (5.41)

$$N = C^{-1}B = C^{-1}(C + O(z^{-1}))A(z) = (I + O(z^{-1}))A(z)$$

as $z \rightarrow \infty$. Also (5.22) is satisfied, since it holds for the first row and the functions f and g are analytic at the branch points, except in the case where Q_ν coincides with the branch point w_1 . In that case, f and g may have a pole at $z_1, \omega z_1, \omega^2 z_1$. However, in that case $N_{1,1}$ and $N_{1,2}$ behave like $(z - \omega^j z_1)^{1/4}$ as $z \rightarrow \omega^j z_1$ for $j = 0, 1, 2$, and then (5.22) still holds.

5.6 Proof of Proposition 4.7

Proof. The RH problem for N is solvable if and only if $n\beta - \nu^* - \frac{\tau}{2\pi i} \log 2$ is not an integer. Then (5.20) gives the solution of the RH problem 4.6 for M and this proves part (a) of Proposition 4.7.

Part (b) follows from the similar statement about N . Alternatively, it can also be deduced by a compactness argument.

For part (c) it remains to show that $M_{n,11}$ is given by (2.36). First from (5.20) and (5.15) we get

$$M_{n,11}(z) = 2^{-2u_1(z^3)} N_{n,11}(z) = 2^{2 \int_{z^3}^{\infty_1} \omega_S} N_{n,11}(z)$$

Since $\psi^*(\omega_S) = \omega_R$ this is also

$$M_{n,11}(z) = 2^{2 \int_z^{\infty_1} \omega_R} N_{n,11}(z).$$

For $N_{n,11}(z)$ we have the expression (5.36) with $j = 1$, δ as in (5.33) and ν as in (5.23). Since $\psi^*(\omega_S)$ we also have

$$\delta = -s_0 - \int_{-A^{1/3}}^{\infty_1} \omega_R,$$

and (see (5.15) for u_1),

$$u_1(z^3) = \int_{\infty_1}^z \omega_R,$$

with integration on the first sheet of \mathcal{R} .

Finally, by (5.31) and (5.32), we have $v_1^{(1/2)}(\infty) = 1$ and

$$\begin{aligned} v_1^{(1/2)}(z^3) &= \frac{\eta_1(z^3)}{(3\eta_1(z^3)^2 - 2z^3\eta_1(z^3) - (1+t)z^3)^{1/2}} \\ &= \frac{\xi_1(z)}{(3\xi_1(z)^2 - 2z^2\xi_1(z) - (1+t)z)^{1/2}} \end{aligned}$$

since $\eta_1(z^3) = z\xi_1(z)$. Combining all this we find (2.36). □

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